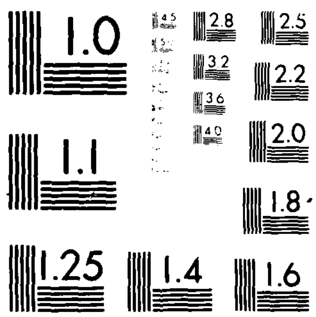
AD-A096 791 NAVAL POSTGRADUATE SCHOOL MONTEREY CA F/G 15/7
ATTRITION MODELLING,(U)
1979 J G TAYLOR MIPR-ARO-22-77

UNCLASSIFIED ARO-14961.7-M NL

1 OF 1

END
DATE
FILMED
4-81
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963 A

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 14961.7-M (18) ARO	2. GOVT ACCESSION NO.: N/A
3. RECIPIENT'S CATALOG NUMBER: N/A	
4. TITLE (and Subtitle): Attrition Modelling	5. TYPE OF REPORT & PERIOD COVERED: REPRINT
	6. PERFORMING ORG. REPORT NUMBER: N/A
7. AUTHOR(s): James G./Taylor	8. CONTRACT OR GRANT NUMBER(s): ARO MIPR 22-77 (15) MIPR-ARO-22-77
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Naval Postgraduate School, Monterey, CA 93940	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: N/A
11. CONTROLLING OFFICE NAME AND ADDRESS: US Army Research Office, PO Box 12211, Research Triangle Park, NC 27709	12. REPORT DATE: (11) 1979
	13. NUMBER OF PAGES: 51
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Submitted for announcement only

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

ARO 14961.7-M

Reprinted with the publisher's permission from pp. 139-189
of Operationsanalytische Spiele für die Verteidigung,
R. K. Huber, K. Niemeyer, and H. W. Hofmann (Editors),
Oldenbourg, München, 1979.

JAMES G. TAYLOR

Attrition Modelling

Abstract This paper surveys current approaches used in the United States for assessing casualties in simulated tactical engagements between general-purpose military forces in conventional air-ground operations. It first discusses the various modelling alternatives available to the military OP worker and then expounds upon both detailed Lanchester-type models of attrition in tactical engagements and also aggregated-force models based on index numbers (e.g. firepower scores). Methodological aspects are emphasized. Simple auxiliary models are used to illustrate modelling points for developing and understanding complex operational models, but examples of current operational models that use these two theoretical approaches of casualty assessment are given. Concerning detailed Lanchester-type models of attrition in tactical engagements, simple auxiliary models are used to illustrate modelling concepts and issues such as (1) various functional forms for attrition rates, (2) determining numerical values for attrition-rate coefficients, (3) various operational factors to be considered in attrition models. Index-number methods for aggregating military capabilities and aggregated-force models of attrition are then discussed, and a Lanchester-type aggregated-force attrition model is developed.

Übersicht Die Arbeit gibt einen Überblick über die derzeit in den Vereinigten Staaten gebräuchlichen Ansätze zur Verlustermittlung in der Simulation taktischer Luft-Landkriegsgefechte und -operationen. Zunächst werden die verschiedenen Alternativen der Modellformulierung angesprochen, die dem Bearbeiter militärischer OR-Probleme zur Verfügung stehen, und dann werden eingehend erörtert sowohl ausführliche Lanchester'sche Modelle von Abnutzungsprozessen in taktischen Gefechten als auch Modelle auf Gesamtstreitkräfte-Ebene, denen Indexzahlen (z.B. Kampfkraftziffern) zugrundeliegen. Dabei werden methodische Aspekte besonders betont. Es wird von einfachen Hilfsmodellen Gebrauch gemacht, um wesentliche Punkte der Modellformulierung beim Entwickeln und Verstehen komplexer Planspiele zu veranschaulichen; es werden aber auch Beispiele gebräuchlicher Planspiele gegeben, die diese zwei theoretischen Zugänge zur

Verlustbewertung benutzen. In Bezug auf ausführliche Lanchester-Modelle von Abnutzungsprozessen in taktischen Gefechten werden einfache Hilfsmodelle verwendet, um Konzepte des Modellansatzes und solche Punkte zu verdeutlichen wie (1) verschiedenartige Funktionsgestalt der Abnutzungsraten, (2) Bestimmung numerischer Werte für die Abnutzungskoeffizienten und (3) Berücksichtigung verschiedener operativer Faktoren im Abnutzungsmodell. Schließlich wird auf Indexzahlmethoden zur Zusammenfassung militärischer Leistungsfähigkeiten und Abnutzungsmodelle auf Gesamtstreitkräfteebene eingegangen, und zuletzt wird ein Lanchester'sches Abnutzungsmodell auf Gesamtstreitkräfteebene entwickelt.

1. Introductory Remarks

The fundamental role of ground-combat troops (in the U.S. Army's own words, e.g. see [1, p. iv]) is to "shoot, move, and communicate." Consequently models of combat operations must in some manner represent the attendant processes of attrition, movement, and C^3 (i.e. command, control, and communications). This paper will focus on the modelling of combat attrition, although some consideration does have to be given to the other two processes of movement and C^3, especially as they influence the attrition process. More specifically, we will examine the principal methodologies used in the United States for assessing casualties in simulated combat engagements, with emphasis on simulated engagements in conventional ground-combat operations.

The two attrition-modelling approaches that we will examine in detail are as follows:

(A1) detailed Lanchester-type[1)] models of attrition in tactical engagements,

(A2) aggregated-force casualty-assessment models based on the use of index numbers to quantify military capabilities.

We will try to be fairly comprehensive in our examination of these attrition-modelling approaches, and when details must be omitted, references to further details in the literature will be given.

1) So-called after the pioneering work of F.W. Lanchester [2]. We will refer to any differential-equation model of combat as a Lanchester-type combat model or as a system of Lanchester-type differential equations (or sometimes simply as Lanchester-type equations). The state variables are typically the numbers of the various different weapon-system types.

Although scientific work on the analysis, modelling, and evaluation of military systems in simulated combat (see Taylor [3] for further details) has been going on in the United States for well over thirty years now, there unfortunately is no single article (or even small number of introductory articles) that is a satisfactory introduction to the topic of attrition modelling, especially as concerns recent developments (and there have been some very important ones) and trends. The author has welcomed the opportunity to try to integrate and synthesize concepts and ideas on attrition modelling. Previously, such ideas and results have been widely scattered in the literature (see, however, survey articles by Dolansky [4] and Taylor [3]; [5]).

Combat (especially combat between company-sized units and larger) is a fantastically complex random process. This inherent complexity of the combat process leads to great complexity in operational models of combat attrition. However, for purposes of understanding the modelling approaches and concepts that may be used to build such operational models, it is convenient to abstract much simpler auxiliary models and to study them.[2)] This idea of using simple auxiliary models to illustrate modelling points for developing and understanding complex operational models apparently has not been clearly articulated within the combat-modelling community (at least in the United States), and therefore before examining currently used attrition-modelling methodologies we will briefly try to establish a conceptual framework for reviewing them.

2. Different Types of Combat Models

Models are representations and idealizations of reality. They have been fairly widely used in the United States as decision aids in defense planning (e.g. to evaluate "on paper" proposed weapon systems during advanced planning). Figure 1 shows us different types of models that have been used to represent combat operations. However, for present purposes let us focus on the three rightmost types of combat models shown in Figure 1 (see Bonder [7] for further details and discussion):

(T1) war games,
(T2) computer simulations,
(T3) analytical models.

---

2) The reverse process of starting with a simple model and then elaborating upon it and enriching it in details is, of course, the approach usually used by model developers to build their models. See W.T. Morris [6] for a lucid discussion of this enrichment process.

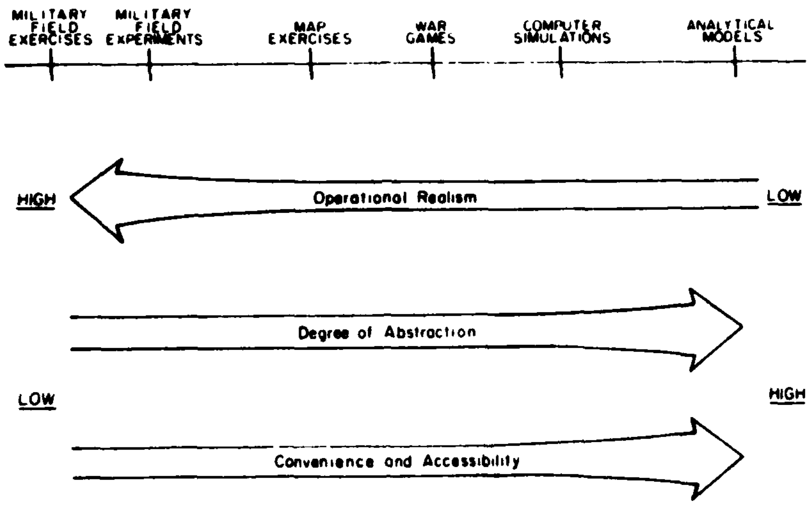


Fig. 1: The Spectrum of Types of Combat Models

We will now briefly discuss each of these different types of combat models.

The distinguishing feature of war games is that they use people playing roles to simulate decision processes, whereas simulations and analytical models use algorithms or some other type of logic to represent decision processes. Simulations "act out" combat operations, usually in great detail. In a sense they recreate the sand table (usually with the help of a digital computer), and battles are acted out on this automated sand table. Such combat simulations usually use so-called pseudo-random numbers to determine the outcomes of random events (e.g. the outcome of firing at a target) and are consequently called Monte Carlo simulations. Analytical models use symbols to represent the combat process (and hence they are really mathematical models). The process under study is analyzed and abstracted (i.e. decomposed into basic events and activities). Then mathematical submodels of events and activities are developed and integrated into an overall structure. Analytical models of any degree of complexity usually do not yield convenient analytical solutions but require numerical approximation methods and a digital computer for the generation of numerical results.

Now that we have discussed the general types of combat models that are available for defense analyses, it is time to focus on the types of models that have been used to assess outcomes (in particular, casualties) of simulated tactical engagements. Thus, the three approaches that are currently used in the United States for assessing casualties in simulation tactical engagements are as follows (see Bonder and Farrell [8] for further details):

(A1) firepower scores (see Stockfisch [9, pp. 6-27]),

(A2) Monte Carlo simulation 10 ; 11 ,

(A3) analytical models (i.e. differential equations) [8].

The firepower-score[3)] approach is an index-number method for aggregating the heterogeneous forces on a side into a single equivalent homogeneous force. We will discuss such models further in Section 6.1 below. Lanchester-type (i.e. differential-equation[4)]) models are the basic type of analytical model used for assessing casualties, although other types of analytical submodels may be used to represent processes such as target acquisition.

If we try to relate each of the above casualty-assessment approaches to a particular scale of combat operations, we find that the firepower-score approach and Monte Carlo simulation are at opposite ends of the spectrum of the scale of combat operations (i.e. the size of the units involved) (see Table I). The contents[5)] of Table I are only generally true, with exceptions certainly existing. As we see from this table, the firepower score approach has been primarily used for casualty assessment in large-scale (i.e. corps-level and

---

3) Indices of the relative combat capabilities of military units (based on a "scoring system" for the weapons employed in the units) have been used by military gamers and force planners in the United States for at least thirty five years. We are here generically referring to both such indices and the associated scores as firepower scores. (See Stockfisch [9] and below in the main text for a discussion of the difference in meaning between the words score and index as generally used in defense analyses.) Members of this family of scores and indices are firepower score/index of combat effectiveness (FS/ICE), firepower potential/unit firepower potential (FP/UFP), firepower potential score/index of firepower potential (FPS/IFP), weapon effectiveness index/weighted unit value (WEI/WUV), weapon effectiveness value (WEV), antipotential potential, etc. (see Stockfisch [9] for further references and a guide to the literature about firepower scores; also see Honig et al. [12, Appendix C to Chapter II]). When two names separated by a "slash") are given above, the first name (e.g. FS) denotes the scoring system for weapon-system types, while the second (e.g. ICE), identifies the index number for a unit's capability.

4) We are calling both differential-equation and also difference-equation models Lanchester-type models. In practice, all operational models of combat systems of any degree of complexity use finite-difference methods for computation and thus are really difference-equation models. However, for purposes of model building, it is much more convenient to think in terms of differential equations.

5) As pointed out by Shubik and Brewer [13], documentation of combat models (when it does exist) is generally poor. The following documentation and information is, however, exceptionally good for this field. General information about contemporary combat models in the United States is available in [14] and [15]. Further information about ATLAS may be found in Kerlin and Cole [16] or [10], while that about CEM may be found in [17] or [18]. Further information about CARMONETTE may be found in Zimmerman [11] or Adams

Table I: Combat-Assessment Approach Related to Scale of Combat Operations

Modelling Approach	Scale of Combat - Example[5] of Model
firepower score	theater - ATLAS, CEM
Monte Carlo Simulation	infantry: platoon - ASARS II armor: company/battalion- DYNTACS, CARMONETTE
Lanchester-type Model	battalion - BONDER/IUA division - DIVOPS theater - VECTOR-2, TWSP, BALFRAM, DMEW

5) See footnote on previous page.

theater-level) combat models. Although there are exceptions, high-resolution Monte Carlo simulation has been a feasible assessment approach only when there have been no more than about 100 elements (e.g. individual tanks, crew-served weapons, etc.) on each side. On the other hand, Lanchester-type models have been developed for the entire spectrum of combat operations, from combat between company/battalion-sized units to theater-level combat operations.

Thus, we find that Monte Carlo simulations have been used to assess casualties in small-unit combat (i.e. combat between battalion-sized units and smaller), while the firepower-score approach applies primarily to large-scale (i.e. corps-level and theater-level) combat. However, Lanchester-type models have been developed in the United States for the full spectrum of combat operations, from small-unit combat to large-scale operations. Thus, if one wants to assess casualties for simulated tactical engagements between battalion-sized units or larger, there are only two types of models for assessing casualties in such tactical engagements:

(1) detailed Lanchester-type models, and

5) cntd.
et al. [19], while that about DYNTACS may be found in [20] or [21]. Information about BONDER/IUA and its various derivative models may be found in [8]; [14]; [22]; [23]; [24], while that about DIVOPS may be found in [25]. The theater-level combat model named VECTOR is documented in [26] and [27]. DMEW (see [28]) is also a theater-level model, as is TWSP (see Fain et al. [29]).

(2) aggregated-force models based on quantifying military capabilities with index numbers (i.e. firepower-score models).

3. Detailed Models versus Aggregated-Force Models of Attrition in Tactical Engagements

Without the modern high-speed digital computer both high-resolution Monte Carlo simulations such as DYNTACS and CARMONETTE and also differential combat models such as BONDER/IUA and its many derivatives would be impossible. The modern computer provides not only large-scale memory capacity but also the ability to perform millions of arithmetic operations per second. The numerical integration of a system of hundreds of ordinary differential equations is consequently possible. Today Lanchester-type complex system models, which rely on modern digital computer technology for their implementation (e.g. see Bonder and Honig [22] or [26]), have been developed for various levels of combat, from combat between battalion-sized (e.g. see Bostwick et al. [30] or Hawkins [23]) and division-sized [25] units to theater-level operations (e.g. see Cordesman [31], Farrell [32], or [26]; [27]).

All the above complex operational models that are conceptually based on Lanchester-type equations, however, model combat attrition in detail and explicitly consider the many different weapon-system types that can be individually attrited. These weapon-system types include different types of weapon systems in maneuver units and different types of fixed-wing aircraft, as well as separately represented field artillery, air defense artillery, and helicopter weapon systems. Such Lanchester-type models represent attrition in a way that reflects the internal dynamics of combat activities and relates these dynamics to specific weapon-system parameters and tactics considered important in small-unit engagements. The effects of individual weapon-system types on the outcome of a theater-level campaign are clearly observable and bear a clear relationship to the input performance assumed (see [27] for further details). We will further consider such models in Section 5 below.

A different approach for modelling attrition in large-scale (i.e. theater-level) combat operations is to represent attrition in a macroscopic fashion. The many different weapon systems on one side are all combined together into a single scalar quantity, the "combat capability" of the force, and combat causes attrition of this index number. The attrition of combat capability is determined with the help of casualty-rate curves that relate the relative combat capabilities of the forces and other tactical factors to their casualty rates (expressed in an aggregated fashion). Losses of individual weapon-system types are then determined by some means of disaggregation. Such aggregated loss-rate relations are apparently largely judgmentally determined (although having some alleged basis in empirical combat data), and the author knows of no conceptual approach or mathematical models for relating weapon-system-performance parameters and other operational variables to the numerical determination of these aggregated-force loss rates.

4. Use of Simple Auxiliary Models for developing and understanding Complex Operational Models

As pointed out above and well known to this audience, combat (especially large-scale combat such as considered for NATO studies, e.g. see Huber et al. [33]) is a very complex process, and consequently the operational models that have been developed in, for example, the United States are also very complex. However, for purposes of discussing and understanding such complex models it is much more convenient to consider simplified versions of them than to consider the complex models themselves.

Thus, one approach for understanding the reasons why a large-scale <u>complex operational model</u> produces certain output results for particular numerical input data is to abstract a simpler model (e.g. one with fewer variables or simpler functional relations between them) from the complex one. This <u>simple auxiliary model</u> is then used to investigate the system dynamics of the more complex model by considering alternative assumptions and data estimates. The simplified auxiliary model should be intuitively plausible and transparent but yet it should capture the basic essence of the complex operational model. This idea of using relatively simple auxiliary models in conjunction with a complex operational model is, of course, not new[6], but the author knows of no clear articulation of this approach for understanding large-scale combat models.

[6] Geoffrion [34] has suggested a similar conceptual approach of using a simple auxiliary model to generate tentative hypotheses to be tested in a full-scale operational model and thus to provide guidance for further (computerized) higher-resolution investigations. We also have felt (Taylor [35]) that the use of relatively simple auxiliary models in conjunction with complex operational models has much to offer for the analysis of military operations (see also Nolan and Sovereign [36] and Weiss [37]).

5. Overview of Detailed Lanchester-Type Models of Attrition in Tactical Engagements

In this section we will review various specific models and associated modelling methodology for detailed Lanchester-type models of attrition in tactical engagements. Such models relate casualties sustained to both the tactical and operational variables of the simulated engagement and also weapon-system capability parameters. We will illustrate most modelling points and concepts by considering models for combat between two homogeneous forces. Complex operational models that have been developed in the United States (see Section 5.6 below) take such simple models as their point of departure and have been built by the process of enrichment (see Footnote 2) and Morris [6] for further details).

The Lanchester-type models that we consider here are all deterministic in the sense that each of them will always yield the same output for a given set of input data. Even though combat between two military forces is a complex random process, such deterministic combat models are commonly used for computational reasons in defense-planning studies, for example, to assess the relative importance of various weapon-system and force-level parameters, since they give essentially the same results for the mean course of combat as do corresponding stochastic attrition models.

5.1 Various Functional Forms for Attrition Rates

Different military/operational situations have been hypothesized to yield different functional forms for Lanchester-type equations. We begin by reviewing two simple differential-equation models that were originally considered by F.W. Lanchester [2]. These models are fundamental for representing attrition and still may be considered to be a point of departure for modelling combat attrition.

Lanchester [2] hypothesized that under conditions "of modern warfare" attrition in combat between two homogeneous forces could be modelled by

$$\begin{cases} \dfrac{dx}{dt} = -ay & \text{with } x(0) = x_0 , \\[2ex] \dfrac{dy}{dt} = -bx & \text{with } y(0) = y_0 , \end{cases} \tag{5.1.1}$$

with $x(t)$ and $y(t)$ denote the numbers of X and Y at time t after the battle begins, and a and b are positive constants that are today called Lanchester attrition-rate coefficients. They represent the effectiveness of each side's fire, i.e. its firepower. We will call (5.1.1) Lanchester's equations for modern warfare. We should, of course, only use these equations when both x and $y > 0$ and, for example, set $dx/dt = 0$ when $x = 0$. Today military operations analysts use (5.1.1) to model combat attrition under the following two different sets of circumstances:

either (a) both sides use "aimed" fire and target-acquisition times are constant, independent of the number of enemy targets (Weiss [38]),

or (b) both sides use "area" fire and a constant-density defense (Brackney [39]).

From (5.1.1) one can readily deduce Lanchester's famous "square law"

$$b(x_0^2 - x^2) = a(y_0^2 - y^2) , \tag{5.1.2}$$

which yields many important results, e.g. X will win a fight-to-the-finish if and only if $x_0/y_0 > \sqrt{a/b}$ (e.g. see Taylor and Comstock [40]).

Lanchester also hypothesized that when both sides use "area" fire, combat attrition could be modelled by

$$\frac{dx}{dt} = -axy , \quad \text{and} \quad \frac{dy}{dt} = -bxy . \tag{5.1.3}$$

It should be noted that the attrition-rate coefficients a and b in (5.1.1) and (5.1.3) represent different physical quantities and are consequently given by different expressions (see Taylor [41]). For want of a better alternative, we will call (5.1.3) Lanchester's equations for area fire, even though they have been hypothesized to apply to certain cases of "aimed" fire (see below). Today analysts use (5.1.3) to model combat attrition under the following two different sets of circumstances:

either (a) both sides use "area" fire and a constant-area defense (Weiss [38], Brackney [39]),

or (b) both sides use "aimed" fire with the rate of target acquisition being inversely proportional to the number of enemy targets and also being the controlling (and constraining) factor in the attrition process (Brackney [39]).

From (5.1.3) one can also deduce Lanchester's famous "linear law"

$$b(x_0 - x) = a(y_0 - y) , \tag{5.1.4}$$

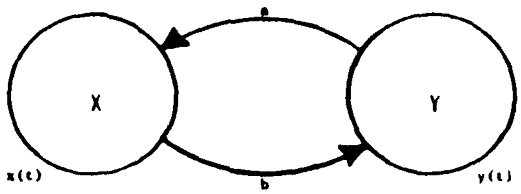
which also yields many important results.

Thus, several different sets of physical assumptions may be hypothesized to yield the same functional form for an attrition rate. Consequently, it is more convenient to refer to a model for combat between two homogeneous forces in terms of the functional forms for the two attrition rates than to refer in terms of the assumptions as we have done above. Let us therefore introduce a very convenient shorthand for referring to such homogeneous-force Lanchester-type combat models. It basically involves using a two-part descriptor X|Y, where X describes the attrition rate for the X force and similarly for Y. X and Y take on their values according to the type of proportionality for the various terms in a side's attrition rate. This proportionality is expressed in terms of the number of firers (denoted as F) and/or the number of targets (denoted as T). If the attrition rate is independent of the numbers of firers and targets, we use the letter C (for constant attrition rate). When there is more than one term in a side's attrition rate, the same approach is applied to each term, with a plus sign separating each component term of the attrition rate.

Let us now consider some examples to illustrate this shorthand. For example, for Lanchester's equations of modern warfare (5.1.1), the X force's attrition rate is (-dx/dt) = ay so that it is proportional to only the number of enemy firers (and similarly for the Y force's attrition rate). Consequently, we will refer to it as a F|F Lanchester-type attrition process (or, simply, F|F attrition). Similarly, Lanchester's equations for area fire (5.1.3) represent FT|FT attrition, since each side's attrition rate is proportional to the product of the numbers of firers and targets.

Figure 2 shows various different attrition-rate functional forms that have been considered in the literature of the Lanchester theory of combat. We have used the above shorthand notation for referring to these various attrition processes in the figure. Also shown for each process are the state equation (if not too complicated) and the first person (known to this author) to have considered it.

Let us now briefly examine various sets of physical assumptions that have been hypothesized to yield the five basic attrition-rate functional forms shown in Figure 2. Conditions hypothesized to yield the F|F and FT|FT attrition processes have been discussed above, while conditions for the F|FT process (equivalently, the FT|F process) are just a combination of these two sets, with one set applying for each side. For example, Brackney 39 has hypothesized that the F|FT attrition process occurs for an assault by the X force against the Y force's defensive positions, in which the defenders use aimed fire (with X targets being readily acquired by virtue of their "assault posture") and so do the attackers, only their search time for Y targets is



ATTRITION PROCESS	DIFFERENTIAL EQUATIONS	STATE EQUATION
F|F	$\frac{dx}{dt} = -ay$ $\frac{dy}{dt} = -bx$	LANCHESTER (1914) $b(x_0^2-x^2) = a(y_0^2-y^2)$ square law
FT|FT	$\frac{dx}{dt} = -axy$ $\frac{dy}{dt} = -bxy$	LANCHESTER (1914) $b(x_0-x) = a(y_0-y)$ linear law
F|FT	$\frac{dx}{dt} = -ay$ $\frac{dy}{dt} = -bxy$	BRACKNEY (1959) $\frac{b}{2}(x_0^2-x^2) = a(y_0-y)$ mixed law
T|T	$\frac{dx}{dt} = -ax$ $\frac{dy}{dt} = -by$	PETERSON (1953) $b \ln \frac{x_0}{x} = a \ln \frac{y_0}{y}$ logarithmic law
(F+T)|(F+T)	$\frac{dx}{dt} = -ay - \beta x$ $\frac{dy}{dt} = -bx - \alpha y$	MORSE and KIMBALL (1951) (generally very complicated)

Fig. 2: Various functional forms for attrition rates that have been considered in the Lanchester-combat-theory literature

relatively large (and inversely proportional to enemy troop density) by virtue of the enemy remaining under cover in their defensive positions. Also Deitchman [42] has used F|FT attrition to model insurgency operations (i.e. guerrilla warfare) in which Y-force guerrillas ambush X-force counterinsurgents. He hypothesized "aimed" fire for the ambushing Y force, which fires on the X force ("caught in the open"), but that the ambushed X force can only return area fire, since its members do not know the exact positions of individual Y ambushers and consequently return fire into only the general area known to be occupied by the enemy.

Peterson [43]; [44] has hypothesized that T|T attrition, i.e.

$$\frac{dx}{dt} = -ax \,, \qquad \text{and} \qquad \frac{dy}{dt} = -by \,, \tag{5.1.5}$$

characterizes the early stages of a small-unit engagement in which the vulnerability of a force dominates its ability to acquire enemy targets. He introduced this model to extend the available choice of basic combat models and also because it does fit limited data for a certain type of engagement, i.e. a tactical situation in which all weapons of the two forces are within effective range of the enemy but when (due to cover, concealment, or expert camouflage) no two weapons are actually intervisible. Once the battle actually begins, however, this model no longer applies. H.K. Weiss [45] has suggested that force vulnerability may become the dominant factor in causing losses as combat units increase in size and become increasingly inefficient. G. Clark [46] has used T|T attrition (5.1.5) for the early stages of a small-unit engagement in his COMAN model.

The last attrition-rate functional form shown in Figure 2 is that of (F + T)|(F + T) attrition, i.e.

$$\frac{dx}{dt} = -ay - \beta x \,, \qquad \text{and} \qquad \frac{dy}{dt} = -bx - \alpha y \,. \tag{5.1.6}$$

Two situations that have been hypothesized to yield the above equations are (see Figure 3):

(S1) F|F attrition in combat between two homogeneous forces with "operational" losses (Bach et al. [47] and Morse and Kimball [48]),

(S2) F|F attrition in combat between two homogeneous primary forces (see Weiss [37]) with superimposed effects of supporting fires not subject to attrition (Taylor and Parry [49]).

In the first situation (S1), for example, the term (βx) in X's loss rate, i.e. ($-dx/dt$), represents "operational" losses, i.e. losses due to causes other than enemy action [47] (e.g. losses due to sickness, accidents, desertions, etc.). In other words, the model holds that a force suffers a certain amount of casualties because of its very size. In the second situation (S2), it is assumed that F|F attrition occurs between the primary fighting forces, e.g. infantries, and that the supporting weapons employ area fire against enemy infantry (again see Figure 3).

A general form for homogeneous-force attrition rates (and which yields the square, linear, and logarithmic laws as special cases) has been given by R. Helmbold [50], who proposed a modification of Lanchester's equations for "modern warfare" to account for inefficiencies of scale for the larger force when force sizes are grossly unequal. Helmbold has emphasized that Lanchester's

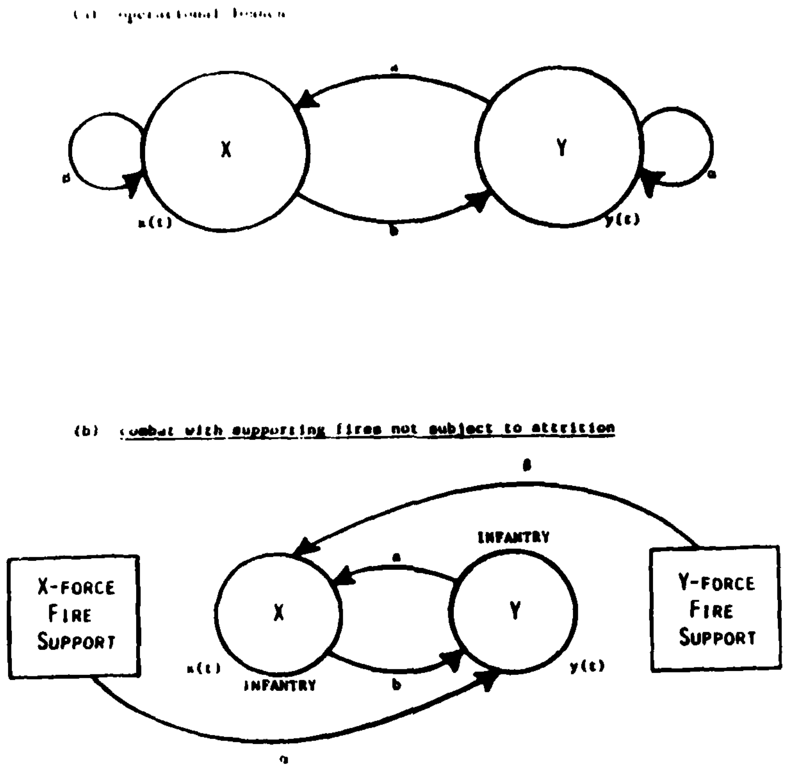


Fig. 3: Two different combat situations that have been hypothesized to yield (F + T)|(F + T) attrition

classic equations for modern warfare (5.1.1), i.e. the F|F attrition model, imply that no matter how unequal the force levels of the opposing combatants may be, the full destructive capability of each side can always be focused on the enemy. However, sheer limitations on available space (to say nothing of terrain-masking and reaction-time effects) may well prevent the larger force from using its full destructive capability against the smaller force, especially if the opposing force is much smaller.

Thus, R. Helmbold [50] was led to hypothesize that the larger force suffers inefficiencies of scale in producing casualties when force sizes are grossly unequal. He confirmed this hypothesis[7] by analyzing historical data (initial

---

7) The empirical basis for Helmbold's proposed modification is not reported in [50], and consequently we have discussed this point here. Apparently his CORG reports [51]; [52]; [53] were not released for general distribution to the public until after the publication of [50]. For this reason we have

and final force levels for each side) for several hundred battles (see Helmbold [51]; [52]; [53]). Using the model (5.1.1), Helmbold computed the combatants' relative fire effectiveness a/b for each battle and found that a/b and the initial force ratio x_0/y_0 were strongly positively correlated (see p. 7 of [51] and pp. 31-35 and 58-59 of [52]). Thus, as the initial force ratio of X to Y increases, the relative fire effectiveness of an individual X combatant to that of an individual Y one decreases. In other words, a side fights less "efficiently" as the initial force ratio (friendly to enemy forces) increases.

Based on such consideration of historical combat data, Helmbold [50] introduced a modification that alters relative force-attrition (or fire-effectiveness) capability by a factor depending on the force ratio. For temporal variations in fire effectiveness, his proposed modification of (5.1.1) would read

$$\begin{cases} \dfrac{dx}{dt} = -a(t)\cdot E_Y(\frac{x}{y})\cdot y & \text{with } x(0) = x_0 , \\ \dfrac{dy}{dt} = -b(t)\cdot E_X(\frac{y}{x})\cdot x & \text{with } y(0) = y_0 , \end{cases} \tag{5.1.7}$$

where a(t) and b(t) denote time-dependent attrition-rate coefficients, and E_X and E_Y denote the fire-effectiveness-modification factors that model the inefficiencies of scale. Helmbold argued that the fire-effectiveness-modification factors should satisfy the following three requirements:

(R1) $E_X(u) = E_Y(u) = E(u)$ (i.e. same inefficiencies of scale for each side),

(R2) $E(u)$ is an increasing function of its argument,

(R3) $E(1) = 1$.

Helmbold then considered the special case in which E(u) is a power function, i.e. $E(u) = u^c$ with $c \geq 0$. In this case, (5.1.7) becomes

$$\begin{cases} \dfrac{dx}{dt} = -a(t)\cdot(\frac{x}{y})^{1-W}\cdot y & \text{with } x(0) = x_0 , \\ \dfrac{dy}{dt} = -b(t)\cdot(\frac{y}{x})^{1-W}\cdot x & \text{with } y(0) = y_0 , \end{cases} \tag{5.1.8}$$

where we will call W the "Weiss parameter" (see [3]). It follows that $W = 1-c$. We will refer to (5.1.8) as the <u>equations for Helmbold-type combat</u>.

---

7) cntd.
expounded upon how Helmbold's empirical investigations [51]; [52]; [53], which were all done well before the submission of the manuscript [50], motivated his subsequent proposed modification in [50].

In the case of constant attrition-rate coefficients, (5.1.8) becomes

$$\begin{cases} \frac{dx}{dt} = -a\cdot(\frac{x}{y})^{1-W}\cdot y & \text{with } x(0) = x_0 , \\ \frac{dy}{dt} = -b\cdot(\frac{y}{x})^{1-W}\cdot x & \text{with } y(0) = y_0 , \end{cases} \tag{5.1.9}$$

where a and b denote constant attrition-rate coefficients. Consequently, we find that $dx/dy = (a/b)(y/x)^{2W-1}$, whence the state equation is given by

$$b(x_0^{2W} - x^{2W}) = a(y_0^{2W} - y^{2W}) \quad \text{for } W \neq 0 , \tag{5.1.10}$$

and

$$b \ \ell n(x_0/x) = a \ \ell n(y_0/y) \quad \text{for } W = 0 . \tag{5.1.11}$$

Thus, for the case of constant attrition-rate coefficients, the equations for Helmbold-type combat yield the square law when $W = 1$, the linear law when $W = 1/2$, and the logarithmic law when $W = 0$. Hence, we should think of (5.1.9) as a general combat model which contains many of the classic homogeneous-force combat models as special cases.

5.2 Fractional Casualty Rates for some Selected Models

Many aggregated-force combat models assess casualties by using a curve (or table) of percent daily casualties versus the force ratio of the attacker's combat power divided by that of the defender (with other tactical variables held constant). In Section 6.3 we will model such a casualty-rate curve with a differential-equation model. Thus, we will find it very instructive for future developments in this paper to examine casualty rates (expressed as a fraction of the side's current strength) for several of the above simple homogeneous-force attrition models.

Let us begin by examining the fractional casualty rate for a side in combat modelled by Lanchester's equations for modern warfare (5.1.1). We therefore consider, for example, X's fractional casualties per unit time. From the first of equations (5.1.1), we obtain

$$\left(-\frac{1}{x}\frac{dx}{dt}\right) = \begin{pmatrix}\text{X's fractional casualties} \\ \text{per unit time}\end{pmatrix} = \frac{a}{u} = av , \tag{5.2.1}$$

where u denotes the force ratio of X's force level to that of Y, i.e. $u = x/y$, and v denotes its reciprocal, i.e. $v = y/x$.

In Figure 4 we have plotted X's fractional casualties per unit time as a

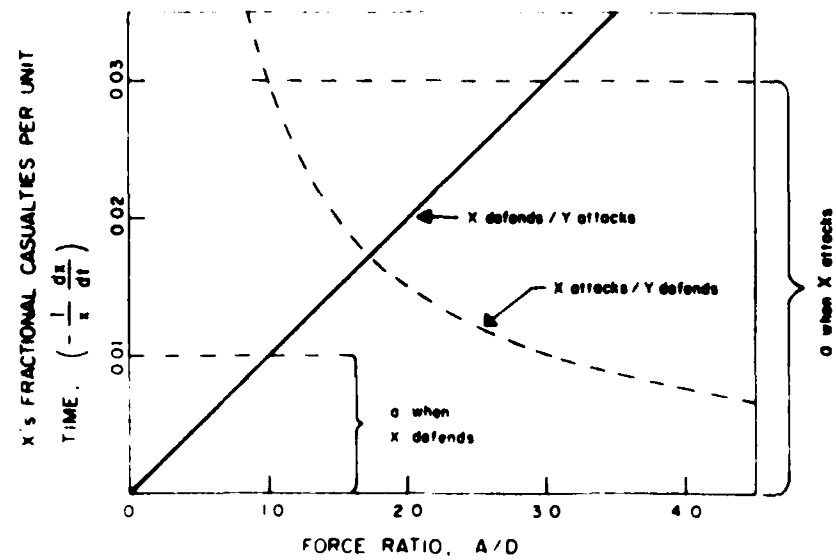


Fig. 4: Relation between X's casualty rate (expressed as a fraction of his current force level x(t)) and the force ratio (expressed as the ratio of the attacker's force level to that of the defender) for Lanchester's classic model of "modern warfare" $\frac{dx}{dt} = -ay$.
[NOTE: In the legend of the above figure, A denotes the attacker's force level, and D denotes that of the defender.]

function of a certain force ratio. The force ratio that we have used is the quotient of the attacker's strength (here, force level) divided by that of the defender and have denoted it as A/D, since most combat analyses use a force ratio of this type. The solid line in Figure 4 represents X's fractional casualties per unit time as a function of the force ratio A/D when X defends and Y attacks. It is a straight line through the origin with a slope equal to the value of the Lanchester attrition-rate coefficient a as the reader can see by referring back to (5.2.1). The value of a may also be obtained from this figure by setting $v = 1$ in (5.2.1) and reading off the corresponding value for X's fractional casualty rate from the curve. Thus, we have developed some important relations between the fractional casualty rate and the Lanchester attrition-rate coefficient. Finally, the dashed line (which is a hyperbola) in Figure 4 represents X's fractional casualties per unit time as a function of the force ratio A/D in the other case in which X attacks and Y defends. Similar curves for daily casualty rates (but not expressed in terms of differential equations) are commonly used to assess casualties in currently operational large-scale aggregated-force combat models (see Section F).

Similarly, for the Helmbold type equations of combat (5.1.8) we have

$$\left(-\frac{1}{x}\frac{dx}{dt}\right) = \begin{pmatrix}\text{X's fractional casualties}\\ \text{per unit time}\end{pmatrix} = \frac{a}{u^W} = av^W . \qquad (5.2.2)$$

In Figure 5 we have similarly plotted X's fractional casualties per unit time versus the force ratio $v = y/x$ (denoted in the figure as A/D) for the case in which Y attacks and X defends. In this figure $W = 1$ corresponds to the case in which X's casualty rate is proportional to only the number of enemy firers, and (in the symmetric case in which Y's casualty rate has the same functional form) consequently the corresponding attrition model is given by Lanchester's equations for modern warfare (5.1.1), which yield the square law. As before, we see that in this case (i.e. when $W = 1$) X's fractional casualties per unit time are directly proportional to the force ratio A/D when Y attacks and X defends. Referring back to the first of equations (5.1.9), we see that $W = W_1$ corresponds to a more efficient use of firepower for force ratios $v = A/D = y/x > 1$ than does $W = W_2$ when $1 \geq W_1 > W_2$, since the corresponding fire-effectiveness-modification factor for $W = W_1$ (i.e. $E_Y(x/y) = (x/y)^{1-W_1}$) is greater than that for $W = W_2$ when $y/x > 1$. Figure 6 shows the same type of plot when X is the attacker and Y is the defender. In this case, the casualty-rate curve corresponding to the square law is a hyperbola.

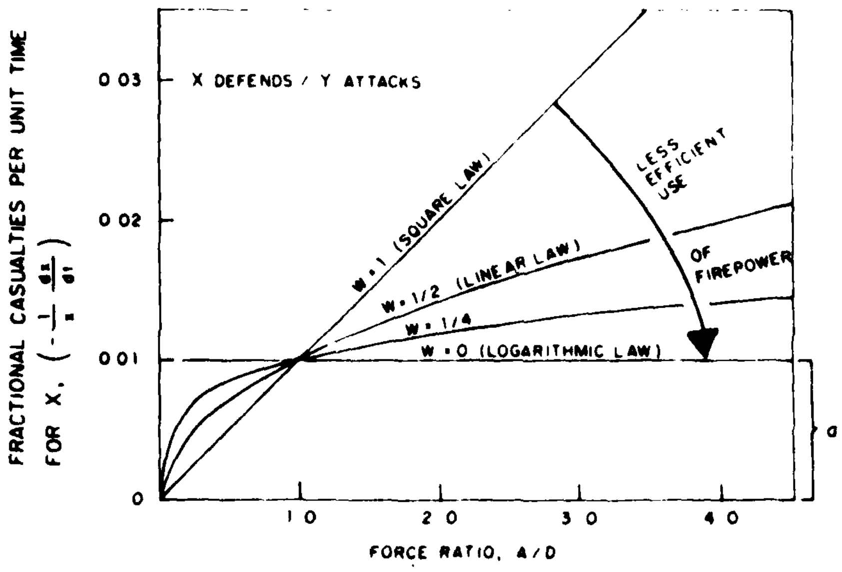


Fig. 5: Relation between X's fractional casualty rate and the force ratio for the model $\frac{dx}{dt} = -a\cdot(\frac{x}{y})^{1-W}\cdot y$ when X defends

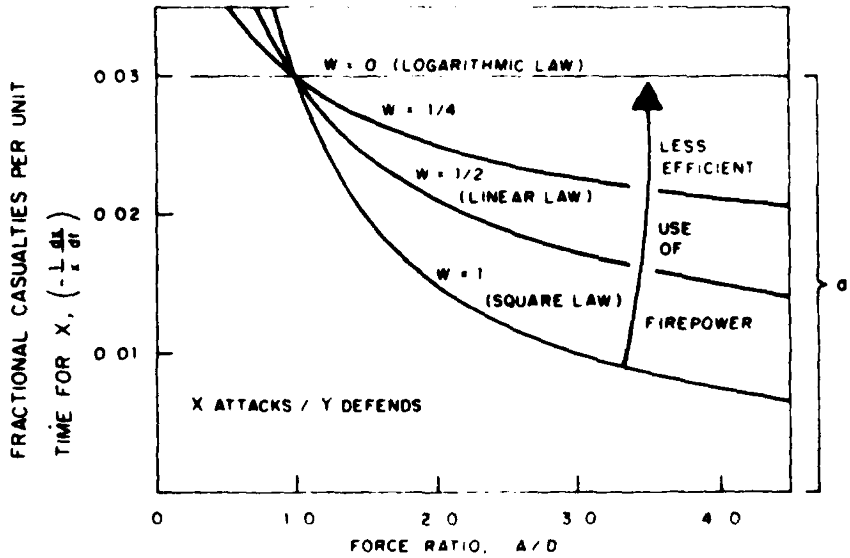


Fig. 6: Relation between X's fractional casualty rate and the force ratio for the model $\frac{dx}{dt} = -a\cdot(\frac{x}{y})^{1-W}\cdot y$ when X attacks

5.3 Determination of Numerical Values for Attrition-Rate Coefficients

For applying any kind of detailed Lanchester-type attrition model to study a particular hypothesized combat engagement in a defense-planning study, one must be able to predict rates at which weapon systems would inflict and sustain casualties. Within the context of an assumed functional form for the attrition rates, this means determining numerical values for the associated Lanchester attrition-rate coefficients. Two approaches that have been used in the United States for such numerical determinations are based on using

(A1) a statistical estimate based on "combat" data generated by a detailed Monte Carlo combat simulation, and

(A2) an analytical submodel of the attrition process for the particular combination of firer and target types.

The first approach (A1) has been called by Bonder [7] the use of a fitted-parameter analytical model, since the basic idea is to statistically estimate parameters for the attrition-rate coefficient from the output of a high-resolution Monte Carlo combat simulation (see Figure 7). The "combat" data or outputs of the simulation are used to fit one or more free parameters in the

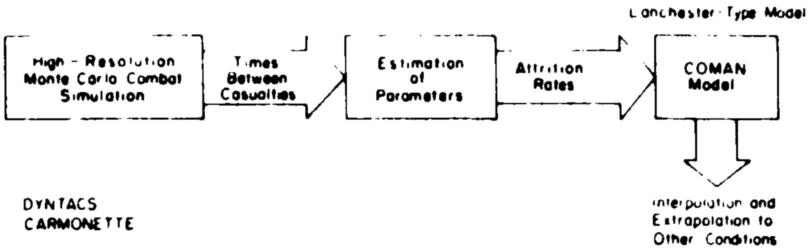


Fig. 7: Basic idea of the fitted-parameter analytical model for complimentary use of Monte Carlo simulation and Lanchester-type attrition models

analytical model so that it will at least duplicate and hopefully predict results comparable to those obtainable from the simulation model. The COMAN model (see Clark [46]) is an example of such a fitted-parameter model. Encouraging results have been reported (see [10]).

S. Bonder [7] has called the second approach (A2) the use of a freestanding or independent analytical model, since this type of analytical model can be run independently of any detailed Monte Carlo simulation of the combat process. The basic conceptual idea is to develop an analytical expression for each attrition-rate coefficient by considering a single firer engaging a "passive" target (i.e. one that does not fire back). One designs such a model to use the same types of inputs as used by detailed Monte Carlo simulations of the same combat process. An example of such an independent analytical model is the BONDER/IUA differential combat model, which was first used in the United States in 1969 [7], and the many subsequently enriched versions of it (see Footnote 5) for references).

Bonder and Farrell [8] have used the second approach (A2) to develop general methodology for determining attrition-rate coefficients for a wide spectrum of weapon-system types engaging specified target types. Basically, their approach is founded upon calculation of a Lanchester attrition-rate coefficient as being the reciprocal of the expected time for an individual firer to kill a single target (see Barfoot [54]). For the model (5.1.1) this means

$$a = \frac{1}{E[T_{XY}]}, \tag{5.3.1}$$

where T_{XY}, a random variable (frequently abbreviated as r.v.), denotes the time required for an individual Y firer to kill a single X target, and $E[T]$ denotes the expected value of T (a r.v.).

To facilitate such analysis Bonder and Farrell [8] have classified the engagement of particular target types by different weapon-system types according to the taxonomy shown in Table II. Weapon-system types are first classified according to the mechanism by which they kill particular target types (i.e. their lethality characteristics) as being either impact-to-kill systems or area-lethality systems. Within each of these two categories Bonder and Farrell further classify weapon-system types according to how they use firing information to control the system's aim point and their delivery characteristics, i.e. the firing doctrine employed. Expressions have been developed for Lanchester attrition-rate coefficients corresponding to the weapon-system classifications tagged with asterisks * in Table II.

Table II: Classification of Weapon-System Types for the Development of Lanchester Attrition-Rate Coefficients for the Model (5.1.1) (from Bonder and Farrell [8])

<u>Lethality Mechanism</u>

(1) Impact

(2) Area

<u>Firing Doctrine</u>

(1) Repeated Single Shot

- (a)* Without Feedback Control of Aim Point
- (b)* With Feedback on Immediately Preceding Round (Markov-Dependent Fire)
- (c) With Complex Feedback

(2) Burst Fire

- (a)* Without Aim Change or Drift in or Between Bursts
- (b)* With Aim Drift in Bursts, Aim Refixed to Original Aim Point for Each Burst
- (c) With Aim Drift, Re-aim Between Bursts

(3) Multiple Tube Firing: Feedback Situations (1a), (1b), (1c)

- (a)* Salvo or Volley

(4) Mixed-Mode Firing

- (a) Adjustment Followed by Multiple Tube Fire
- (b)* Adjustment Followed by Burst Fire.

*Indicates that analysis of this category has been performed by Bonder and Farrell [8].

Let us now illustrate Bonder's attrition-rate-coefficient methodology for the model (5.1.1). A large class of weapon systems (e.g. tanks firing at tanks, anti-tank weapon systems firing at tanks, etc.) may be classified as Markov-dependent-fire weapons, i.e. the outcome of the firing of a round by the weapon system depends on only the outcome of the immediately preceding round. For such weapon systems and an impact-to-kill lethality machanism, Bonder [55]; [56] (see also Kimbleton [57]) has developed the following expression for the expected time for an individual firer to kill an enemy target

$$E[T] = t_a + t_1 - t_h + \frac{t_h + t_f}{P(K|H)}$$

$$+ \frac{(t_m + t_f)}{P(h|m)} \left\{ \frac{1 - P(h|h)}{P(K|H)} + P(h|h) - p \right\}, \qquad (5.3.2)$$

where all variables are defined in Table III. The corresponding Lanchester attrition-rate coefficient is then given by the reciprocal of (5.3.2), e.g. see (5.3.1) above. The above ideas are the conceptual basis for all attrition calculations in the BONDER/IUA model. Let us finally record here that the precise conditions for (5.3.2) to hold are essentzilly given by the following assumptions:

(A1) Markov-dependent fire with parameters p, $P(h|h)$, and $P(h|m)$,

(A2) geometric distribution for the number of hits required for a kill with parameter $P(K|H)$.

Although (5.3.2) is a rather formidable-looking expression, it is easily evaluated with an automated algorithm such as is available for a computerized model such as the BONDER/IUA. Moreover, in several special cases of interest (5.3.2) simplifies appreciably. In the simplest case, we have (1) target-acquisition time negligible (i.e. set $t_a = 0$), (2) uniform rate of fire (i.e. $t_1 = t_h = t_m = 1/\nu$, where ν is simply the firing rate), (3) statistical independence among firing outcomes (i.e. $p = P(h|h) = P(h|m) = P_{SSH}$), and (4) negligible time-of-flight of the projectile (i.e. set $t_f = 0$); and then (5.3.2) reduces to

$$E[T] = 1/(\nu\, P_{SSK}), \qquad (5.3.3)$$

where the single-shot kill probability P_{SSK} is given by $P_{SSK} = P_{SSH} \cdot P(K|H)$. In this case, the Lanchester attrition-rate coefficient is, for example, consequently given by

$$a = \nu_Y P_{SSK_{XY}}, \qquad (5.3.4)$$

Table III: Factors Included in Expression for Lanchester Attrition-Rate Coefficient for Single-Shot Markov-Dependent-Fire Weapon Systems with a Geometric Distribution for the Number of Hits Required for a Kill

Time to acquire a target, t_a
Time to fire first round after target acquired, t_1
Time to fire a round following a hit, t_h
Time to fire a round following a miss, t_m
Time of flight of the projectile, t_f
Probability of a hit on first round, p
Probability of a hit on a round following a hit, $P(h\|h)$
Probability of a hit on a round following a miss, $P(h\|m)$
Probability of destroying a target given it is hit, $P(K\|H)$

which is a very intuitively appealing result.

It is very instructive to combine an idea due to Brackney [39] concerning the *target-acquisition* process with Bonder's general expression (5.3.2). In this case, we assume that (1) the mean time to acquire a target is inversely proportional (let k denote the constant of proportionality) to target density, and also (2) through (4) above in the previous paragraph. Then, for example, the expected time to kill a target reduces to

$$E[T_{SY}] = \frac{k_Y A_X}{x} + \frac{1}{\nu_Y P_{SSK_{XY}}} , \qquad (5.3.5)$$

where A_X denotes the "presented" area occupied by the X force and which is visually searched by a Y firer. When the first term predominates (i.e. the constraining factor in killing targets is acquiring them), we may set $E[T_{XY}] = k_Y A_X/x$ and obtain

$$\frac{dx}{dt} = -\tilde{a}xy , \qquad (5.3.6)$$

where $\tilde{a} = 1/(k_Y A_X)$. Thus (as pointed out by Brackney [39]), the assault of a defensive position may be more appropriately modelled with the equations of a FT|F attrition process.

As seen in (5.3.1) and (5.3.2), the Lanchester attrition-rate coefficient depends directly on the factors given in Table III. However, it also depends indirectly on variables such as the range between firer and target, target posture, terrain, target motion, etc. Many people (e.g. Bonder and Farrell [8]) feel that for many tactical situations the principal factor is the range between firer and target. For illustrative purposes, let us therefore consider that the Lanchester attrition-rate coefficients in (5.1.1) explicitly depend on range,[8] i.e.

$$a = \alpha(r), \quad \text{and} \quad b = \beta(r), \tag{5.3.7}$$

where r denotes the range between firer and target. Bonder and Farrell [8, pp. 196-200] have used expressions like (5.3.1) and (5.3.2) to examine the variations in weapon-system kill rate with range. The following functional form fits data for a number of representatives weapon systems (see Bonder and Farrell [8] for further details)

$$\alpha(r) = \begin{cases} \alpha_0 \left(1 - \dfrac{r}{r_\alpha}\right)^{\mu} & \text{for } 0 \le r \le r_\alpha, \\ 0 & \text{for } r \ge r_\alpha, \end{cases} \tag{5.3.8}$$

where r_α denotes the maximum effective range of Y's weapon system, $\alpha_0 > 0$, and $\mu \ge 0$. The constant μ allows us to model the range dependence of the weapon system's kill rate (see Figure 8).

When a weapon system employs "area" fire and enemy targets defend a constant area, the expression for the Lanchester attrition-rate coefficient takes a different form and also depends (among other things) on the vulnerable area of the target (denoted as a_V) and the lethal area of the projectile fired by the firer's weapon system (denoted as a_L). In general a rather complicated expression is obtained for such an attrition-rate coefficient (e.g. see Bonder and Farrell [8, pp. 141-162]), but in special cases these simplify (cf. equation (5.3.4) above), e.g. for "small arms fire" when $a_V >> a_L$ and for a "weapon of great lethality" when $a_L >> a_V$.

Thus, two cases in which simple expressions are obtained for attrition-rate coefficients for "area" fire and a constant-area defense are as follows for:

(1) small arms fire (i.e. $a_V >> a_L$), and

(2) weapons of large lethality (i.e. $a_L >> a_V$).

---

8) In actual application, a model like BONDER/IUA or VECTOR-1 allows such attrition-rate coefficients to additionally depend on other variables (e.g. target posture, terrain, etc.) that depend on the positions of firers and targets and other operational factors and that may change over time.

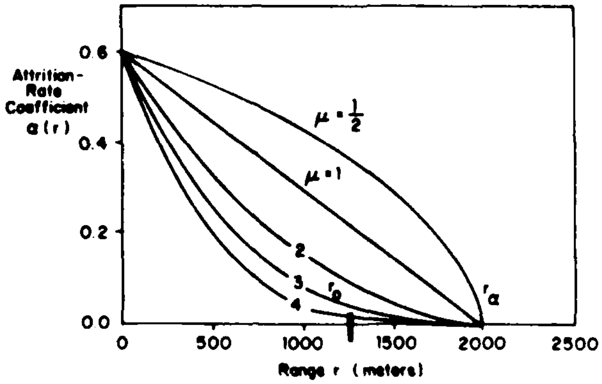


Figure 8: Dependence of Y's attrition-rate coefficient $\alpha(r)$ on the exponent μ with the maximum effective range of the weapon system and kill rate at zero range held constant. (NOTES: 1. The maximum effective range of the system is denoted as r = 2000 meters. 2. $\alpha(0) = \alpha_0 = 0.6$ X casualties/(unit time × number of Y firers) denotes the weapon-system kill rate for Y at zero force separation (range). 3. The opening range of battle (see Section 6.4) is denoted as r_0 = 1250 meters and (as shown) $r_0 < r_\alpha$.)

In both cases, we may consider the assumptions of "area" fire and a constant-area defense by targets to yield

$$\frac{dx}{dt} = -\nu_Y P_{SSK_{XY}} Y. \tag{5.3.9}$$

For small arms fire, we may calculate $P_{SSK_{XY}}$ by consider a lethal "dot" being randomly placed into a large region (of area A_X) that contains x "vulnerable circles" (each of area a_{V_X}) randomly placed in the region. It follows that $P_{SSK_{XY}} = a_{V_X} x/A_X$ and thus

$$\frac{dx}{dt} = -\frac{\nu_Y a_{V_X}}{A_X} xy. \tag{5.3.10}$$

For weapons of large lethality (e.g. artillery), we may calculate $P_{SSK_{XY}}$ by considering a "lethal circle" being placed into the region which contains x "vulnerable dots". If there are no multiple kills, then $P_{SSK_{XY}} = a_{L_Y} s/A_X$ and thus

$$\frac{dx}{dt} = -\frac{\nu_Y a_{L_Y}}{A_X} xy, \tag{5.3.11}$$

where a_{L_Y} denotes the lethal area of the projectile fired by Y's weapon system. A more refined analysis is contained in Bonder and Farrell [8] (see also Cherry [58]).

5.4 Additional Operational Factors to be considered in Attrition Models

In order to reflect more of the complexity of combat we can enrich the above simple attrition models by considering additional operational factors such as:

(F1) range-dependent weapon-system capabilities,
(F2) other temporal variations in fire effectiveness,
(F3) target-acquisition considerations,
(F4) command, control, and communications,
(F5) unit breakpoints,
(F6) unit deterioration due to attrition,
(F7) suppressive effects of weapon systems,
(F8) effects of logistics constraints.

The addition of such factors to attrition models greatly enhances their operational realism. Although we will not in all cases be able to report full details of computational results, it turns out that such enriched models usually yield quite different results than, for example, Lanchester's classic constant-coefficient equations for a F|F attrition process.

S. Bonder [59]; [60] (see also Bonder and Farrell [8]) has stressed the importance for evaluating many types of proposed weapon systems of using variable-coefficient combat models to represent temporal variations in firepower on the battlefield. Such a case occurs, for example, when the range between firers and targets changes appreciably during battle. We have already pointed out above (see Section 5.3) that attrition-rate coefficients should generally be considered to depend on firer-target range.

Let us therefore consider "aimed-fire" combat between two homogeneous forces and assume that target-acquisition times do not depend on the numbers of targets. We further assume that one force attacks at constant speed the other force's static defensive position. Then (for x and $y > 0$) we have

$$\begin{cases} \dfrac{dx}{dt} = -\alpha(r)y & \text{with } x(t=0) = x_0 \, , \\[2ex] \dfrac{dy}{dt} = -\beta(r)x & \text{with } y(t=0) = y_0 \, , \end{cases} \tag{5.4.1}$$

where r denotes the range between opposing forces (see Figure 9), and $\alpha(r)$ and $\beta(r)$ denote range-dependent attrition-rate coefficients. Range is related to time by

$$r(t) = r_0 - vt , \tag{5.4.2}$$

where r_0 denotes the opening range of battle and $v > 0$ denotes the constant attack speed.

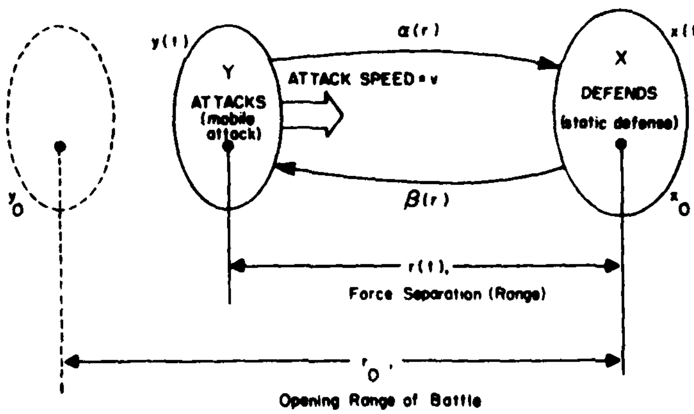


Figure 9: Diagram of Bonder's constant-speed attack model. Force separation, $r(t)$, is given by $r(t) = r_0 - vt$.

Let us now illustrate the importance of such range-dependent weapon-system capabilities. We assume that the Y weapon system's kill rate $\alpha(r)$ is given by (5.3.8) and similarly for $\beta(r)$ with parameters r_β and ν. In Figure 10 we have plotted force-level trajectories for three different battles, denoted as (A), (B) and (C). In all these battles both weapon systems have the same maximum effective range, i.e. $r_\alpha = r_\beta = r_e$, and the battle begins at this range, i.e. $r_0 = r_e$. For these battles we have held the kill rates at zero force separation, i.e. $\alpha_0 = \alpha(0)$ and β_0, constant and have varied between battles the manner in which $\alpha(r)$ and $\beta(r)$ depend on range, i.e. for $0 \le r \le r_e$

(A) constant-constant: $\alpha(r) = \alpha_0$ and $\beta(r) = \beta_0$,

(B) linear-linear: $\alpha(r) = \alpha_0(1-r/r_e)$ and $\beta(r) = \beta_0(1-r/r_e)$,

(C) linear-quadratic: $\alpha(r) = \alpha_0(1-r/r_e)$ and $\beta(r) = \beta_0(1-r/r_e)^2$.

We see from Figure 10 that battle outcome may be quite sensitive to the

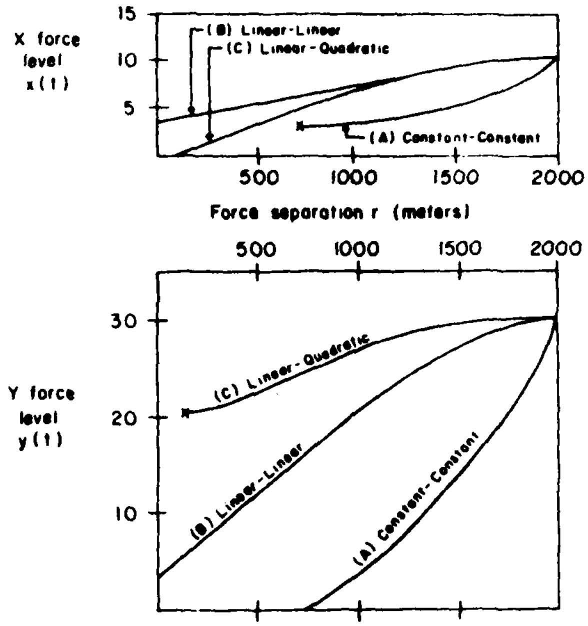


Figure 10: Force-level trajectories of X and Y forces for three different battles denoted in the figure as (A), (B) and (C) and explained in the main text with each side's fire effectiveness modelled by the power attrition-rate coefficients for $r_0 = r_\alpha = r_\beta = r_e$ = 2000 meters, α_0 = 0.06 X (casualties/minute) per Y firer, β_0 = 0.6 Y (casualties/minute) per X firer, v = 5 mph, x_0 = 10, and y_0 = 30. The symbol × denotes the end of a force-level trajectory due to annihilation of the enemy force.

variation in weapon-system kill rate with range. This type of battle was first studied by Bonder [59]; [60], and it also appears as an example in much of the author's work (see Taylor [3]; [41]; [61], Taylor and Brown [62]; [63], and Taylor and Comstock [40]).

If we use (5.4.2) to eliminate range from the attrition-rate coefficients in (5.4.1) we obtain the following model with time as the independent variable (see Taylor [3]; [41])

$$\begin{cases} \dfrac{dx}{dt} = -a(t)y & \text{with } x(0) = x_0 \,, \\[2ex] \dfrac{dy}{dt} = -b(t)x & \text{with } y(0) = y_0 \,, \end{cases} \tag{5.4.3}$$

where the time-dependent attrition-rate coefficients are given by

$$a(t) = k_a(t + C)^{\nu}, \quad \text{and} \quad b(t) = k_b(t + C + D)^{\mu}, \tag{5.4.4}$$

with $C = (r_\alpha - r_0)/v$ and $D = (r_\beta - r_\alpha)/v$. Moreover, we should use a model like (5.4.3) instead of (5.1.1) when there are temporal variations in fire-power due to changes over time in firing rates, target-acquisition rates, target postures and hence vulnerabilities, etc. [i.e. the inputs for calculating the attrition-rate coefficients change over time; see (5.3.1) and (5.3.2)]. An operational model such as BONDER/IUA actually recomputes such attrition-rate coefficients at every "time step" (used for numerical integration of the system of differential equations for casualty assessment) during the battle.

Another important factor to consider is engagement termination, i.e. when to break off the engagement and stop casualties. Although there is some conflicting evidence (see Helmbold [64]), most military planners in the United States usually assume that a ground-combat unit will break off an engagement and try to disengage from the enemy when it has suffered a certain percentage of casualties for a company-sized unit in the attack and fifty percent in the defense). Thus, the simplest model of engagement termination is to assume that battle outcome depends deterministically on only the force levels: the first side to be reduced to a previously specified force level (his "breakpoint") is assumed to break off the engagement (he "loses"). Incorporating this idea into the simple model (5.1.1), we obtain

$$\begin{cases} \dfrac{dx}{dt} = \begin{cases} -ay & \text{for } x > x_{BP} \text{ and } y > y_{BP}, \\ 0 & \text{otherwise,} \end{cases} \\ \\ \dfrac{dy}{dt} = \begin{cases} -bx & \text{for } x > x_{BP} \text{ and } y > y_{BP}, \\ 0 & \text{otherwise,} \end{cases} \end{cases} \tag{5.4.5}$$

where x_{BP} denotes X's breakpoint force level and similarly for y_{BP}.

The inclusion of unit "breakpoints" in attrition models is very important, since battle outcome (i.e. who "wins" and who "loses") is very sensitive to the choice of breakpoints. To illustrate this point, let us observe that Y will "win" an engagement modelled by (5.4.5) if and only if

$$\frac{x_0}{y_0} < \sqrt{\frac{a}{b}\left\{\frac{1 - (f_{BP}^{Y})^2}{1 - (f_{BP}^{X})^2}\right\}}, \tag{5.4.6}$$

where $x_{BP} = f_{BP}^{X} x_0$ and f_{BP}^{Y} is similarly defined. Table IV shows us the sensitivity of battle outcome to the selection of breakpoints. Clearly, more research (especially on empirical considerations) is necessary on various aspects of modelling the engagement-termination process.

Table IV: Influence of Unit Breakpoints on the Outcome of Battle for an Attack by X Against Y with Battle Dynamics Given by Lanchester's Equations for Modern Warfare

Case	$\frac{x_0}{y_0}$	$\frac{a}{b}$	f_{BP}^{X}	f_{BP}^{Y}	$\sqrt{\frac{a}{b}\left\{\frac{1-(f_{BP}^{Y})^2}{1-(f_{BP}^{X})^2}\right\}}$	WINNER	$\frac{x_f}{x_0}$	$\frac{y_f}{x_0}$
1	3,0	5,0	0,8	0,5	3,23	Y	0,8	0,59
2	3,0	5,0	0,7	0,5	2,71	X	0,76	0,50
3	3,0	5,0	$f_{BP}^{Y} = f_{BP}^{X}$		2,24	X	$> f_{BP}^{X}$	f_{BP}^{Y}

NOTE X is the Attacker

As we saw in Section 5.3, target acquisition is reflected in the simple model for the Lanchester attrition-rate coefficient through t_a, which appears in (5.3.1) through (5.3.2) see also the "Brackney model" (5.3.5) . However, Vector Research, Inc. (see [26, pp. 103-108] or [27, pp. 43-45]) has developed a more refined (i.e. enriched in operational details) model for the target-acquisition process and its impact on attrition-rate coefficients. They consider that the two major factors determining the value of an attrition-rate coefficient are (1) the acquisition and selection of targets, and (2) the conditional kill rate (i.e. the rate at which acquired targets are destroyed). Concerning target acquisition and selection, the proportion of time that a weapon is actively engaging an enemy target depends on the interaction of three processes:

(P1) the line-of-sight process (which determines when a given target is visible or invisible to a potential firer),

(P2) the target-acquisition process (which determines the time required for a firer to acquire a particular target), and

(P3) the target selection process (which specifies a scheme by which a weapon crew chooses to engage a particular target from among those that have been acquired).

The exact way in which the above three processes interact depends in an essential way on which of two kinds of acquisition and target-selection modes the weapon systems employ - serial or parallel acquisition (see [26] or [27] for further details). Suppressive effects of weapon systems may be accommodated in Vector Research's model, although the phenomenological basis of suppressive effects is poorly understood at this time (see [65]).

5.5 Modelling Attrition for Combat between Heterogeneous Forces

We have considered above various aspects of attrition modelling for combat between two homogeneous forces, but actual combat consists of many different weapon-system types operating together as "combined-arms teams". For example, there may be infantry (armed with several types of weapons), tanks, artillery, mortars, etc. on each side. Let us therefore consider combat between such heterogeneous forces and briefly indicate how the above basic ideas on modelling combat attrition are extended and adapted to such cases.

For illustrative purposes, we consider an engagement with m different types of weapon systems on the X side and n for Y (see Figure 11). Although more complicated types of force interactions may be postulated, we will consider the "natural" extension of (5.1.1) to this combat situation. We accordingly assume that

(A1) the attrition effects of various different enemy weapon-system types against a particular friendly target type are additive (no mutual support, i.e. no synergistic effects), and

(A2) the loss rate to each enemy weapon-system type is proportional to the number of enemy firers of that type.

Let Y_{ij} denote those Y_j who engage X_i, and let y_{ij} denote the corresponding number of Y_{ij} and similarly for y_j. Similar quantities are analogously defined for the X force. We observe that we then have

$$y_j = \sum_{i=1}^{m} y_{ij}. \qquad (5.5.1)$$

We now introduce the allocation factor $\psi_{ij} = y_{ij}/y_j$ = fraction of Y_j who engage X_i. It follows that

$$y_{ij} = \psi_{ij} y_j. \qquad (5.5.2)$$

Let a_{ij} denote the "inherent" weapon-system kill rate of Y_j against live X_i targets, i.e. the rate at which one Y_j can kill X_i targets.

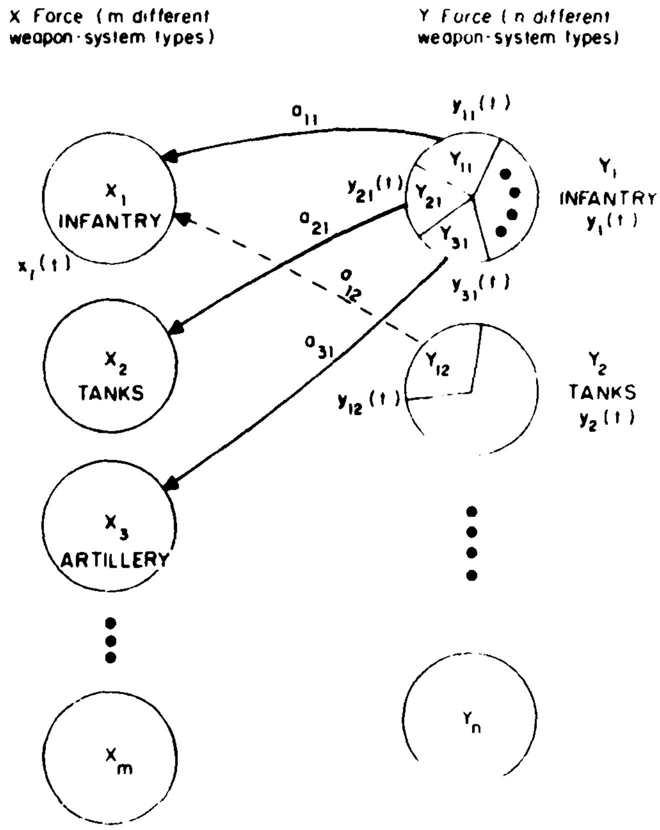


Figure 11: Schematic of combat between heterogeneous forces. In this figure Y_{ij} denotes those Y_j who are engaging X_i, and y_{ij} denotes the corresponding number of Y_{ij} and similarly for y_j. Also, a_{ij} denotes the "inherent" weapon-system kill rate of one Y_j against live X_i targets, i.e. the rate at which one Y_j can kill X_i targets

Let us now examine how (A1) and (A2) lead to the following linear model (no synergistic effects for weapon systems in joint operations) for x_i and $y_j > 0$ for all $i = 1,\ldots,m$ and $j = 1,\ldots,n$

$$\begin{cases} \dfrac{dx_i}{dt} = -\sum_{j=1}^{n} \psi_{ij} a_{ij} y_j \quad \text{with } x_i(0) = x_i^0 \text{ for } i = 1,\ldots,m, \\ \\ \dfrac{dy_j}{dt} = -\sum_{i=1}^{m} \phi_{ij} b_{ij} x_i \quad \text{with } y_j(0) = y_j^0 \text{ for } j = 1,\ldots,n. \end{cases} \tag{5.5.3}$$

Let us now develop (5.5.3) from assumptions (A1) and (A2) above. Assumption (A1) may be stated in mathematical terms as, for example,

$$\frac{dx_i}{dt} = -\sum_{j=1}^{n} \begin{pmatrix} X_i \text{ loss rate} \\ \text{due to } Y_j \end{pmatrix} , \tag{5.5.4}$$

while assumption (A2) means that

$$\begin{pmatrix} X_i \text{ loss rate} \\ \text{due to } Y_j \end{pmatrix} = a_{ij}Y_{ij} = a_{ij}\psi_{ij}y_j, \tag{5.5.5}$$

whence follows (5.5.3) from combination with (5.5.4). If we "absorb" the allocation factors into the attrition-rate coefficients, e.g. let $A_{ij} = \psi_{ij}a_{ij}$, then our linear combat model (5.5.3) may be written as (for x_i and $y_j > 0$ for $i = 1,\dots,m$ and $j = 1,\dots,n$)

$$\begin{cases} \dfrac{dx_i}{dt} = -\displaystyle\sum_{j=1}^{n} A_{ij}y_j \quad \text{with } x_i(0) = x_i^0 \text{ for } i = 1,\dots,m , \\ \dfrac{dy_j}{dt} = -\displaystyle\sum_{i=1}^{m} B_{ij}x_i \quad \text{with } y_j(0) = y_j^0 \text{ for } j = 1,\dots,n . \end{cases} \tag{5.5.6}$$

Although it is an easy matter to develop an analytical solution to (5.1.1) and more difficult to solve (5.4.3) with, for example, the particular coefficients (5.4.4), it is essentially impossible to solve differential equations like (5.5.6) for combat interactions with any degree of complexity. Consequently, numerical integration methods are usually used to numerically determine the force levels as a function of time $x_i(t)$ and $y_j(t)$ in complex operational models like BONDER/IUA. In such complex operational models,[9] the attrition-rate coefficients A_{ij} and B_{ij} are (as they are in the real world) complex functions of the weapon-system capabilities, target characteristics, distribution of the targets, allocation procedures for assigning weapons to targets, etc. These models then attempt to reflect these complexities by partitioning the attrition process into four distinct subprocesses:

(1) the fire effectiveness of weapon-system types firing on live targets,

(2) the allocation process of assigning weapons to targets,

9) Essentially all complex operational Lanchester-type combat models that represent engagements in detail (i.e. do not aggregate forces with firepower scores) and are in current operational use in the United States have been developed by the principals of Vector Research, Inc. The discussion here follows that of Bonder and Farrell [8, pp. 11-17].

(3) the inefficiency of fire when weapon-system types engage other than live targets, and

(4) the effects of terrain on limiting firing activities of weapon-system types and on mobility of the systems.

Bonder and Farrell [8, pp. 16-17] include the effects of the first three subprocesses above on the attrition-rate coefficient, for example, as

$$A_{ij}(r) = \psi_{ij} I^{Y}_{ij} a_{ij}(r) \quad , \tag{5.5.7}$$

where ψ_{ij} denotes the allocation factor (the fraction of Y_j who are firing at X_i), I^{Y}_{ij} denotes the intelligence factor (the fraction of Y_{ij} who are actually engaging live X_i targets), and $a_{ij}(r)$ denotes the "inherent" weapon-system kill rate (the rate at which one Y_j kills live X_i targets when it is firing at them). Here, for simplicity, we have assumed that the inherent weapon-system-kill capability (as quantified by a_{ij}) depends on only the range between firer and target (see Bonder and Farrell [8] for further details). Similar to the case of homogeneous forces, the "inherent" weapon-system kill rate a_{ij} is computed as

$$a_{ij} = 1/E\, T_{X_i Y_j} \quad , \tag{5.5.8}$$

where $T_{X_i Y_j}$ (a r.v.) denotes the time for a single Y_j firer to kill an X_i target.

Thus, Bonder and Farrell's [8] approach (see also [26]; [27] and Cherry [58]) basically decomposes the battlefield into unit engagements, and these are further decomposed into a series of one-on-one duels between opposing weapon-system types. For each firer-target pair one must perform a detailed analysis of a single firer engaging a passive target. Force interactions are then tied together with attrition equations similar to (5.5.6), and these assessment equations are made to respond to the evolution of combat (e.g. changing firer positions) through the operational factors influencing kill rates. Terrain effects are incorporated into such models by computing intervisibility (i.e. existence of line-of-sight) for each target-firer pair based on their map locations. Consideration is given to cover, concealment, terrain roughness, etc. but time does not allow us to go into further details here.

5.6 Current Detailed Models of Tactical Engagements

The following are currently operational models (used in the United States) that use detailed Lanchester-type equations to assess casualties in tactical engagements:[10)]

battalion-level combat: BONDER/IUA, BONDER AIRCAV, BLDM, AMSWAG, FAST,
division-level combat: DIVOPS,
theater-level combat: VECTOR-1.

The modern large-scale digital computer has made such large-scale models possible. The data and data-base problems associated with such models are, however, formidable although no less so than those for detailed Monte Carlo combat simulations. For example, VECTOR-1 may require between 200,000 and 300,000 pieces of input data for a "typical" run. Such models consider heterogeneous forces, battle plans (ground order of battle and air order of battle), target acquisition, allocation of fire, fire support by ground weapons, movement, intelligence, command and control, logistics, etc. They have been developed, though, from the basic analytical structure discussed above by the process of enrichment (which we have also considered above).

Although VECTOR-1 may well be the prototype of the theater-level combat model of the future, as of August 1977 it apparently had not been used operationally [14], whereas it was estimated that (for the same time period) the approximate frequency of use of, for example, ATLAS was 600 times per year and that of CEM was 25 times per year (see [14] for further details). Even BLDM was only used 4 to 5 times per year [14]. A partial explanation of the relatively low frequency of use of such operational differential combat models may be that they are rather demanding in resources (especially highly technically qualified people to maintain, modify, and exercise them).

6. Overview of Aggregated-Force Models of Attrition in Tactical Engagements

In stark contrast to the detailed Lanchester-type models of attrition in tactical engagements are the aggregated-force-attrition models that combine all the various different weapon-system types on a side in some particular geographical combat area (or "sector") into a single equivalent homogeneous force.

---

10) Documentation on these models is discussed in Footnote 5); see also Bostwick et al. [30], Cordesman [31], and Farrell [32].

The daily loss in combat power as quantified by the unit's firepower index is then assessed on the basis of several operational factors, principal of which is *the force ratio* (actually the ratio of the attacker's firepower index to that of the defender). Current theater-level combat models typically use curves of daily fractional (or percentage) casualties versus the force ratio (for both the attacker and also the defender for each of several engagement types such as meeting engagement, attack of prepared position, etc.) for this assessment of losses. These curves supposedly have an empirical basis (see [1, pp. 23-28] or Anderson et al. [66, p. 53]; however, Cockrell and Ball [67, especially p. 1-2] have a different opinion). Unfortunately, there is no explicit relationship between weapon-system parameters, operational factors, and attrition as there is for detailed Lanchester-type models (e.g. recall (5.1.1), (5.3.1), and (5.3.2) above; also see [26] or [27, pp. 3-4]).

Although such aggregated-force models are much simpler than the detailed differential combat models and therefore more computationally convenient, a large-scale digital computer is still required for their implementation. Such aggregated-force models have been fairly widely criticized (see, for example, Bonder [7], Honig et al. [12], or Stockfisch [9]), but large-scale conventional-force ground-combat models that use such aggregation techniques have *been and continue* to be essentially the only analysis tools used for large-scale conventional-force military analyses in the United States (see [14]) and also NATO countries [33]. The simple fact is that some type of aggregation must be done in order to model theater-level combat.

6.1 Aggregation of Forces in Combat Analyses

The modern battlefield contains many diverse weapon-system types that complement each other and operate as "combined-arms teams". For example, there can be both mounted and dismounted infantry, tanks, various types of anti-tank weapon systems, artillery, mortars, infantry with rifles, infantry with machine guns, etc. One must then either model such operations in great detail or find some means for aggregating forces. Military planners[11)] and military

11) Military planners have apparently used the firepower-score approach (see below in main text) for at least thirty years (see Mulholland and Specht [68]) to plan operations and to plan and control tactical exercises. Although the origins of using firepower scores for these purposes are somewhat obscure, they are still in use today (see [69]). Furthermore, it appears as though such use of firepower scores in planning was the origin of their use by operations researchers in modelling large-scale ground combat.

operations analysts have consequently developed various index-number approaches for aggregating the diverse combat capabilities of such a heterogeneous military force into a single scalar measure of combat power. Although there are many such indices[12)] of the relative combat capabilities of military units, all[13)] are essentially variations on the same theme, and consequently we will generically refer to any such index-number approach as a firepower-score approach.

The firepower-score approach develops one single number (referred to as the <u>firepower index</u>) to represent the "combat potential" of a military unit. A linear model is used to develop this index number, i.e. the firepower index, from the scores of individual weapon systems as Table V shows. As Stockfisch [9] has emphasized, however, the words <u>score</u> and <u>index</u> should not be regarded as being synonymous. We should use the term <u>firepower score</u> to refer to the military capability or value of a specific weapon system and use the term <u>firepower index</u> - which is obtained by summing scores - to refer to the military capability or value of some aggregation of diverse weapons. In other words, the firepower index of the X force, denoted as I_X, is given by

$$I_X = \sum_{i=1}^{n} s_i x_i ,$$

where s_i denotes the firepower score of the <u>ith</u> X system and x_i denotes the number effective in the unit (see Table V again).

Although many firepower-score methods claim that the firepower score of a weapon system is determined as the product of a measure of single-round lethality and the expected expenditure of ammunition during a fixed period of time, in actuality varying amounts of subjectivity are involved in the development of such a firepower score. For this and other reasons (e.g. see [12]), the firepower-score approach has received a fair amount of criticism. Nevertheless, it is essentially the only approach that has been used to model large-scale combat in currently operational ground-combat models (e.g. see

---

12) Examples of such scores/indices are given in Footnote 3) above. Bode [70] has given an excellent discussion of the use of such index numbers in general-purpose force analysis, while Aldrich and Bode [71] have given a lucid discussion of the conceptual problems of aggregation in theater-level combat models.

13) The one exception is the antipotential potential or WEV (see Footnote 3) above, Howes and Thrall [72], and Anderson [73]; [74]), which may be exercised in the running of IDAGAM (see Anderson et al. [66]). ATLAS and other models that employ the firepower-score approach, however, are currently much more widely used in the United States than IDAGAM (see [14]).

Table V: Hypothetical Example of Determination of Firepower Index for a Combat Unit

Weapon	Number	Firepower Score	Total Contribution to Firepower Index
Rifle, M-16, 5.56mm	6,000	1	6,000
MG, M-60, .30 cal	150	6	900
MG, M-2, .50 cal	250	10	2,500
Mortar, M-125, 81mm	50	20	1,000
Howitzer, M-109(SP), 155mm	50	40	2,000
Howitzer, 8"	8	30	240
Tank, M60A2	200	100	20,000
		TOTAL FIREPOWER INDEX	32,640

Firepower Index for U. S. Army's 7th Infantry Division

[14]). In other words, unless one duplicates large-scale combat in detail, one must use some type of index-number approach to aggregate the many different types of forces involved in modern large-scale military operations (see last paragraph of Section 5.6). Thus, although it has received varying amounts of criticism from different sources, the firepower-score approach is used by essentially all currently operational large-scale ground-combat models.

In large-scale (i.e. division-level and above) ground-combat models, firepower indices are used as a surrogate for unit strength to:[14)]

(1) determine engagement outcomes,

(2) assess casualties, and

(3) determine FEBA movement.

---

14) Many times the first assessment (i.e. determination of engagement outcome) is omitted. For example, ATLAS and IDAGAM only do the last two assessments. However, some models (e.g. Theater Battle Model (TBM-68) [1]) determine the outcome of an engagement (e.g. whether or not an attack is successful) before assessing casualties. In this case, the casualty-assessment curves depend on the engagement's outcome (see Figures 4 through 7 of [1]).

The force ratio is a major factor (but not the only one) used to make such assessments. Here, however, the term force ratio means the ratio of the attacker's firepower index to that of the defender. Consider, for example, the 7th Infantry Division of the U.S. Army and assume that the firepower scores and other data shown in Table V apply. Then the 7th Infantry Division would have a firepower index of 32,640. If an attacking enemy army group were to have a firepower index of 146,880, then we would have a force ratio of 4.5 (A/D), where A refers to the attacker and D to the defender.

6.2 General Mathematical Structure of Attrition Calculations in Aggregated-Force Models

The usual approach (e.g. see [10]) for assessing casualties in firepower-score-based combat models is to have daily casualties (i.e. the casualty rates) depend directly on the following two factors:

(F1) the force ratio, and

(F2) the engagement type.

It will be instructive for us to hold the last factor constant and further examine how casualty assessment depends on the firepower scores and indices.

The basic mathematical structure of the attrition calculation in aggregated-force models may be thought of as being done in two steps and may be explained as follows:

STEP (I)
(Aggregation of Forces)

$$\begin{cases} x_0 = \sum_{i=1}^{n_X} s_i^X x_i^0 , \\ \\ y_0 = \sum_{i=1}^{n_Y} s_i^Y y_i^0 , \end{cases} \qquad (6\ 2.1)$$

STEP (II)
(Mutual Attrition of the Aggregated Forces)

$$\begin{cases} \left(-\frac{1}{x}\frac{dx}{dt}\right) = A\left(\frac{x}{y}\right) & \text{with } x(0) = x_0, \\ \\ \left(-\frac{1}{y}\frac{dy}{dt}\right) = B\left(\frac{y}{x}\right) & \text{with } y(0) = x_0, \end{cases} \qquad (6.2.2)$$

where s_i^X denotes the firepower score of the ith X weapon-system type, x_i^0 denotes the initial number of the ith X system, x_0 denotes the initial value of the firepower index for the X force, $x(t)$ denotes its value at time t, $A(x/y)$

denotes a given function of the force ratio, t = 0 denotes the start of the attrition calculation, and similarly for the corresponding Y quantities. This calculation is then repeated for each "sector" on the battlefield. Thus, casualties in terms of a loss in the force's combat power are computed from some expression like (6.2.2). In other words, we only know how much the force's combat power was reduced by a day of combat action, and losses of individual component weapon-system types must be obtained by some means of disaggregation.

ATLAS basically computes casualties in the above manner, with the firepower scores (i.e. s_i^X and s_i^Y) being held constant over time. However, IDAGAM dynamically recomputes weapons' values, which correspond to the firepower scores s_i^X and s_i^Y above, according to the antipotential-potential (or eigenvector) method (see Howes and Thrall [72] or Anderson [73]; [74]). The latter calculation involves the numbers of enemy targets, allocations of friendly fire, and kill probabilities against enemy targets.

We have given the basic structure for attrition calculations in aggregated-force models above. In actual application such models give attention to a multitude of details on combat operations, e.g. positioning of units, logistics considerations, allocation of fire (especially supporting fires), air defense, air operations including allocation of aircraft to tactical missions, unit breakpoints, terrain factors, intelligence, command and control, order of battle, etc. (e.g. see documentation on CEM [17]; [18] or IDAGAM [66] for further details). Such operational and tactical factors influence exactly how (6.2.1) is computed.

6.3 Fitting a Differential-Equation Model to Loss-Rate Curves typically used to Model Large-Scale Ground Combat Attrition

In this section we will develop a general attrition model, whose general form fits the shape of most loss-rate curves typically used to model large-scale ground combat.[15] All currently operational large-scale combat models in one

15) Examples of such casualty-rate curves may be found in the documentation for the following large-scale ground-combat models (see also Footnote 5) above): ATLAS [10]; [16], CEM [17]; [18], TBM-68 [1], and TAGS [75]; [76]. See [12] for a general discussion about such large-scale models (but for the period before 1971). Although IDAGAM does not use firepower scores (see Footnote 3) above), it uses the same casualty-rate curves as ATLAS (see [66, p. 53]). In fact, it is stated on p. 53 of [66] that until better historical data is available, the standard functional relationships (used in ATLAS) between force ratios and percent casualties must still be used. Finally, models used for NATO planning also employ the firepower-

way or another assess casualties for each side by using such a loss-rate curve consisting of casualty rate (expressed as a fraction or percentage of current strength lost per unit time) plotted against the force ratio. Here, as above, the term force ratio means the ratio of the firepower index of the attacker to that of the defender, denoted as A/D. Also, loss here means loss of value for the side's firepower index, which can then be disaggregated into losses in numbers of different weapon-system types.

In other words, the firepower-score approach takes each side's heterogeneous forces and converts them into an equivalent homogeneous force quantified in terms of a firepower index, daily reduction in each side's capability (expressed as a reduction in firepower index) is then determined from the ratio of the two such firepower indices, and finally casualties (i.e. losses in numbers of the different weapon-system types) are assessed by some means of disaggregation. We will now discuss how a relatively simple pair of differential equations may be used to model this process and fit these loss-rate curves.

Let us first, however, slightly modify the equations for Helmbold-type combat (5.1.8) by adding terms for "operational" losses, i.e. losses not due to enemy action (e.g. losses due to sickness, accidents, etc.; see Taylor and Parry [49, pp. 523-524]). If we add terms for such operational losses, then equations (5.1.8) become

$$\begin{cases} \dfrac{dx}{dt} = -a(t)\cdot\left(\dfrac{x}{y}\right)^{1-W_Y}\cdot y - \beta(t)x \quad \text{with } x(0) = x_0 , \\[2ex] \dfrac{dy}{dt} = -b(t)\cdot\left(\dfrac{y}{x}\right)^{1-W_X}\cdot x - \alpha(t)y \quad \text{with } y(0) = y_0 . \end{cases} \tag{6.3.1}$$

In equations (6.3.1) we have also added the feature of giving each side its own Weiss parameter. In other words, the firepower-modification factors E_X and E_Y are no longer necessarily the same for both sides, i.e. $E_Y(u;W_Y) = u^{1-W_Y} \neq E_X(u;W_X) = u^{1-W_X}$.

For the case of constant attrition-rate coefficients, (6.3.1) becomes

15) cntd.
score approach and similar casualty-rate curves (e.g. see [33, pp. 287-298]).

$$\begin{cases} \dfrac{dx}{dt} = -a\cdot\left(\dfrac{x}{y}\right)^{1-d}\cdot y - \beta x & \text{with } x(0) = x_0 , \\ \\ \dfrac{dy}{dt} = -b\cdot\left(\dfrac{y}{x}\right)^{1-e}\cdot x - \alpha y & \text{with } y(0) = y_0 , \end{cases} \qquad (6.3.2)$$

where for notational convenience we have denoted W_Y simply as d and W_X as e. For our model (6.3.2), for example, X's fractional casualties per unit time are now given by

$$\left(-\frac{1}{x}\frac{dx}{dt}\right) = \left(\begin{matrix}\text{X's fractional casualties}\\ \text{per unit time}\end{matrix}\right)$$

$$= au^{-d} + \beta = av^{d} + \beta . \qquad (6.3.3)$$

In Figure 12 we show the relation between X's fractional casualties per unit time and the force ratio v = y/x for the case in which X defends (cf. Figure 5). Figure 13 shows the same type of relation when X attacks.

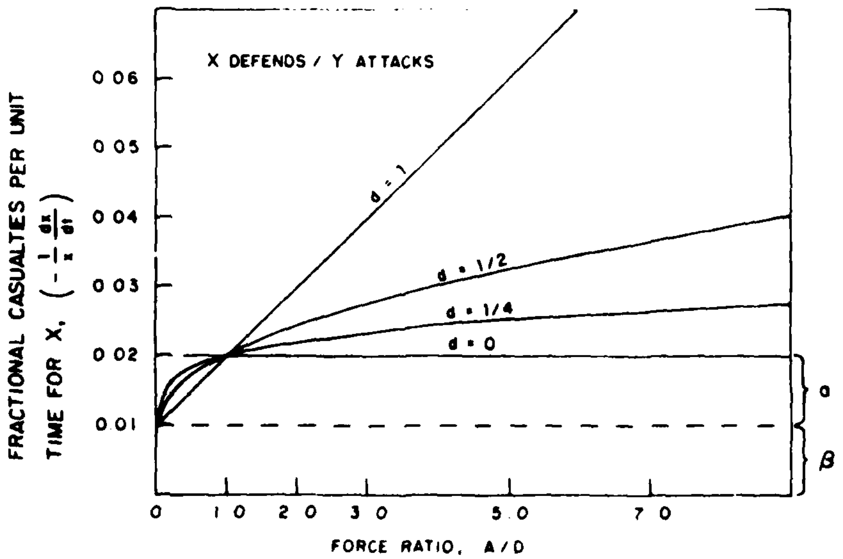


Figure 12: Relation between X's fractional casualty rate and the force ratio for the model $\frac{dx}{dt} = -a\cdot\left(\frac{x}{y}\right)^{1-d}\cdot y - \beta x$ when X defends

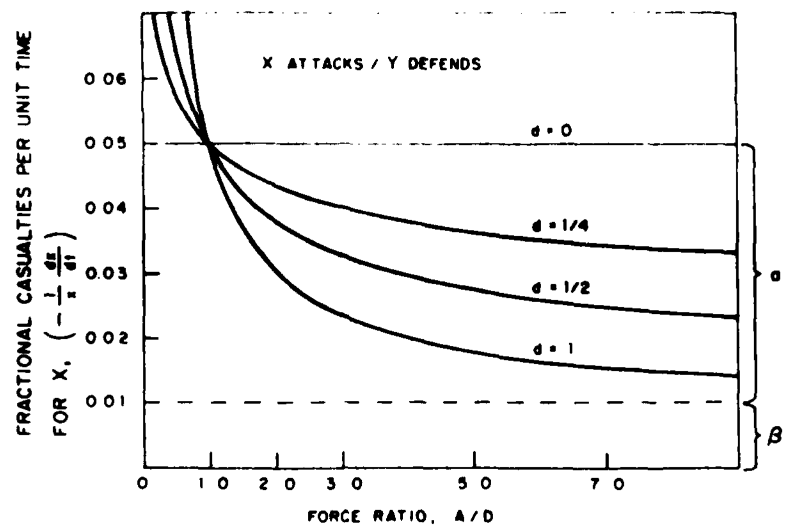


Figure 13: Relation between X's fractional casualty rate and the force ratio for the model
$$\frac{dx}{dt} = -a\cdot\left(\frac{x}{y}\right)^{1-d}\cdot y - \beta x \text{ when X attacks}$$

Essentially all of the principal large-scale ground-combat models currently in operational use in the world today[16] assess casualties using the firepower-score concept and (in one form or another) casualty-rate curves of the form shown in Figure 14, which is taken from documentation on ATLAS [10]. Such casualty-rate curves are typically plots of fractional casualties per unit time (or its equivalent) versus the force ratio (A/D) for different engagement types.[17] Thus, two such plots like those shown in Figure 14 are used to assess casualties, one curve for the attacker and one curve for the defender. It turns out now that the Helmbold-type model (6.3.3) gives a remarkably good fit to almost all these casualty rate curves, i.e. compare Figures 12 and 13 with Figure 14 (i.e. Figure 6-6 on p. 6-5 of [10]), Figure 3 on p. 12 of [75], or pp. 28-31 of [76].

In other words, if (for a given engagement type) we assume that the fractional casualty rate depends on only the force ratio, then the so-called [77] asymptotic-power form (6.3.3) gives a very good fit to most such casualty-rate

---

16) See Footnote 14) and also Footnote 5).

17) For example, as shown in Figure 14, ATLAS [10] distinguishes between seven different types of engagements.

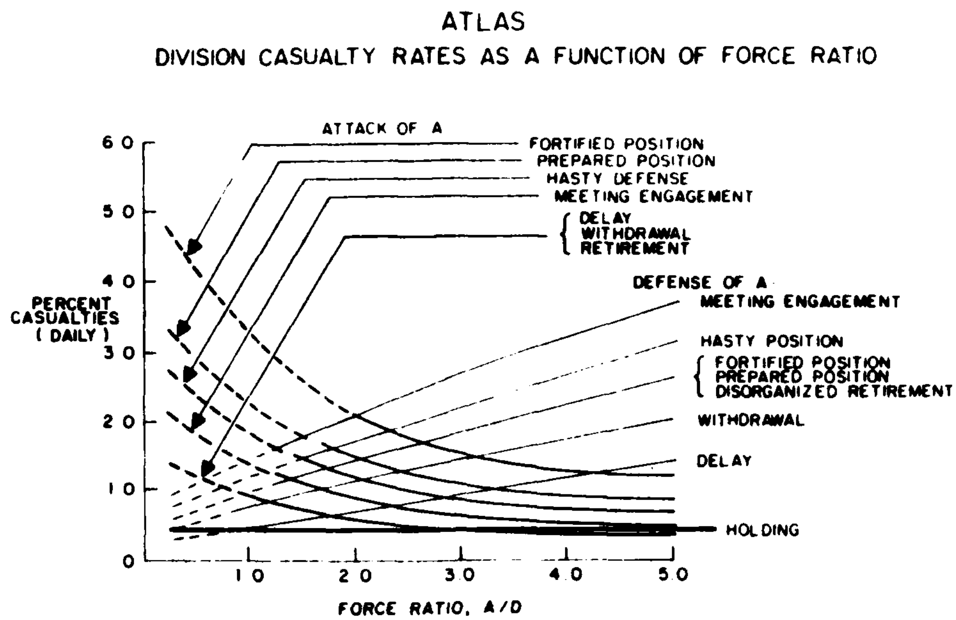


Figure 14: Typical casualty-rate curves used in ATLAS (from [10])

curves currently used, and thus the Helmbold-type equations (6.3.2) may be considered to model the attrition process, with the parameters a, b, α, β, d, and e depending on the type of engagement. Moreover, there are even computerized routines available for the least-squares estimation of these parameters (e.g. see [77] especially Figure 1 on p. 6).

As we discussed in Section 5.2 above, the model (6.3.3), equivalently (6.3.2), can accommodate a wide variety of classic attrition-rate forms, and furthermore a variety of attrition-rate forms have indeed been used in large-scale ground-combat models over the years. For example, ground-combat attrition in the original version of TAGS was assumed to follow the logarithmic law (see [78, p. 29]), cf. d = e = 0 in (6.3.2). Today, attrition is usually modelled as being "intermediate" between the logarithmic and square laws. For example, comparing Figure 12 above to Figure 14 (i.e. Figure 6-6 of [10]), we find that the casualty rate for a defending force is best fit by d near 1 (i.e. $dx/dt = -ay - \beta x$). However, comparing Figure 13 above to Figure 14 we find that a value for d around 1/2 seems more reasonable for the attrition-rate of an attacking force (i.e. $dx/dt = -ax^{1/2}y^{1/2} - \beta x$). All these attrition-rate functional forms may, of course, be handled by the Helmbold-type equations of warfare with operational losses (6.3.2) by taking the appropriate values for the fire-effectiveness-modification exponents d and e. Thus, this general model (6.3.2) has the flexibility of fitting a wide variety of attrition-rate forms that have been used to model large-scale ground combat.

Let us finally note here that the author knows of no acknowledgment of the possibility that the casualty-rate curves such as we have been discussing could be fit by a differential-equation model, or might even have arisen from a formal or informal understanding of simple differential equations. Thus, we have developed an important simplified analytical model of large-unit attrition.

6.4 Current Aggregated-Force Models of Large-Scale Tactical Engagements

The following are currently operational theater-level combat models that use the firepower-score approach to aggregate forces for assessing casualties in the manner discussed above:[18)]

TAGS,
ATLAS,
CEM, and
IDAGAM.

These are essentially the only operational models currently available in the United States for analyzing simulated theater-level combat. It was estimated [14] that as of August 1977 the approximate frequency of use of ATLAS was 600 times per year, that of CEM was 25, and that of IDAGAM II was between 150 and 200.

7. Final Remarks

This paper has attempted to survey approaches for attrition modelling used today in the United States. A major problem is that very little of this work is ever documented (see Shubik and Brewer [13]) (let alone published in the open literature). Consequently, many of the most important conceptual modelling issues are either not well articulated or never explicitly stated at all.

18) References to documentation about these models may be found in Footnotes 5) and 15) above.

The author believes that this seminar of the Hochschule der Bundeswehr München is an important step in improving communications among OR workers interested in combat-modelling methodologies.

Acknowledgments:

The writing of this paper was supported by Code 431 (Naval Analysis Programs) of the Office of Naval Research of the U.S. Navy. A portion of the results reported here, however, were obtained while the author was under contract (R&D Project No. 1L161102H57-05 Math) with the U.S. Army Research Office (Durham, North Carolina), but most of the author's research on attrition modelling has been supported over a number of years by the Office of Naval Research (both through direct funding by Code 431 and also through the Foundation Research Program at the Naval Postgraduate School).

References:

[1] Research Analysis Corporation: Theater Battle Model (TBM-68), Volume VII, Technical Report. R-36, McLean, Virginia, January 1968 (AD 755 534)

[2] Lanchester, F.W.: Aircraft in Warfare: The Dawn of the Fourth Arm - No. V, The Principle of Concentration. Engineering 98, 422-423 (1914) (reprinted on pp. 2138-2148 of The World of Mathematics, Vol. IV, J. Newman (Editor), Simon and Schuster, New York, 1956)

[3] Taylor, J.G.: Recent Developments in the Lanchester Theory of Combat. To appear in Operational Research '78, Proceedings of the Eighth IFORS International Conference on Operational Research, K.B. Haley (Editor)

[4] Dolansky, L.: Present State of the Lanchester Theory of Combat. Opns. Res. 12, 344-358 (1964).

[5] Taylor, J.G.: A Tutorial on Lanchester-Type Models of Warfare. pp. 39-63 in Proceedings of the 35th Military Operations Research Symposium, 1975

[6] Morris, W.T.: On the Art of Modelling. Management Sci. 13, B-707 - B-717 (1967)

[7] Bonder, S.: An Overview of Land Battle Modelling in the U.S. pp. 73-88 in Proceedings of the Thirteenth Annual U.S. Army Operations Research Symposium, Fort Lee, Virginia, 1974

[8] Bonder, S. and Farrell, R.L. (Editors): Development of Models for Defense Systems Planning. Report No. SRL 2147 TR 70-2 (U), Systems Research Laboratory, The University of Michigan, Ann Arbor, Michigan, September 1970, (AD 714 677)

[9] Stockfisch, J.A.: Models, Data, and War: A Critique of the Study of Conventional Forces. R-1526-PR, The RAND Corporation, Santa Monica, California, March 1975

[10] General Research Corporation: A Hierarchy of Combat Analysis Models. McLean, Virginia, January 1973

[11] Zimmerman, R.E.: Simulation of Tactical War Games. pp. 711-762 in Operations Research and Systems Engineering, C. Flagle, W. Higgins, and R. Roy (Editors), Johns Hopkins Press, Baltimore, 1960

[12] Honig, J., Blum, R., Holland, H., Howes, D., Lester, D., Myers, K., and Zimmerman, R.: Review of Selected Army Models. Assistant Vice Chief of Staff (Army), Washington, D.C., May 1971, (AD 887 175)

[13] Shubik, M. and Brewer, G.D.: Models, Simulations, and Games - A Survey. R-1060-ARPA/RC, The RAND Corporation, Santa Monica, California, May 1972

[14] Studies, Analysis, and Gaming Agency, Organization of the Joint Chiefs of Staff: Catalog of War Gaming and Military Simulation Models (7th Edition). SAGA-180-77, Washington, D.C., August 1977

[15] U.S. Army Concepts Analysis Agency: Tabulation of Models of Interest to CAA. Bethesda, Maryland, January 1977

[16] Kerlin, E.P. and Cole, R.H.: ATLAS: A Tactical, Logistical, and Air Simulation. RAC-TP-338, Research Analysis Corporation, McLean, Virginia, April 1969

[17] Bruner, J.A. and 11 other authors: Theater Force Performance in CONAF II, Volume 1 - Methodology, Part 1. OAD-CR-1, General Research Corporation, McLean, Virginia, May 1973

[18] Louer, P.E., Forrester, R.E., Parker, R.W., Shepherd, J.E., Tunstall, J.E., and Willyard, H.A.: Conceptual Design for the Army in the Field Alternative Force Evaluation, CONAF Evaluation Model IV, Part 1 - Model Description. OAD-CR-60, General Research Corporation, McLean, Virginia, September 1974

[19] Adams, H., Forrester, R., Kraft, J., and Oosterhout, B.: CARMONETTE: A Computer-Played Combat Simulation. ORO-T-389, Operations Research Office, The Johns Hopkins University, Chevy Chase, Maryland, February 1961

[20] Bishop, A., and Clark, G.: The Tank Weapon System - Management Summary. Report No. RF 573 AR 69-4 (U), Systems Research Group, The Ohio State University, Columbus, Ohio, June 1969

[21] Clark, G., Parry, S., Hutcherson, D., and Rheinfrank, J.: Small Unit Combat Simulation (DYNTACS X). Report No. RF 2978-FR 71 (U), Systems Research Group, The Ohio State University, Columbus, Ohio, March 1971

[22] Bonder, S., and Honig, J.: An Analytic Model of Ground Combat: Design and Application. pp. 319-394 in Proceedings of the Tenth Annual U.S. Army Operations Research Symposium, Durham, North Carolina, 1971

[23] Hawkins, J.: The AMSAA War Game (AMSWAG) Computer Combat Simulation. AMSAA Tech. Report No. 169, U.S. Army Materiel Systems Analysis Activity, Aberdeen Proving Ground, Maryland, July 1976

[24] Vector Research, Inc.: Analytic Models of Air Cavalry Combat Operations. Report No. SAG-1 FR 73-1, Ann Arbor, Michigan, May 1973

[25] Vector Research, Inc.: DIVOPS: A Division-Level Combined Arms Engagement Model. Report No. ARAFCAS-1 TR-74, Vol. I-II, Ann Arbor, Michigan, 1974

[26] Vector Research, Inc.: VECTOR-1, The Theater Battle Model. WSEG Report 251, Vol. I and II, Ann Arbor, Michigan, July 1974

[27] National Military Command System Support Center: VECTOR-1 System for Simulation of Theater-Level Combat. TM 109-75, Washington, D.C., December 1975

[28] U.S. Army Air Defense School: Deterministic Mix Evaluation World-wide (DMEW), Documentation of DMEW 1.4, Part I (Methodology). Fort Bliss, Texas, September 1973

[29] Fain, W., Fain, J., and Karr, H.: A Tactical Warfare Simulation Program. Naval Res. Log. Quart. 13, 413-436 (1966)

[30] Bostwick, S., Brandi, F., Burnham, D., and Hurt, J.: The Interface Between DYNTACS-X and Bonder-IUA. pp. 494-502 in Proceedings of the Thirteenth Annual U.S. Army Operations Research Symposium, Fort Lee, Virginia, 1974

[31] Cordesman, A. (Editor): Developments in Theater Level War Games. Unpublished materials for C-5 Working Group of 35th Military Operations Research Symposium, 1975

[32] Farrell, R.L.: VECTOR-1 and BATTLE: Two Versions of a High-Resolution Ground and Air Theater Campaign Model. pp. 233-241 in Military Strategy and Tactics, R.K. Huber, L.F. Jones, and E. Reine (Editors), Plenum Press, New York, 1975

[33] Huber, R.K., Jones, L.F., and Reine, E. (Editors): Military Strategy and Tactics. Plenum Press, New York, 1975

[34] Geoffrion, A.M.: The Purpose of Mathematical Programming is Insight, Not Numbers. INTERFACES 7, No. 1, 81-92 (1976)

[35] Taylor, J.G.: Lanchester-Type Models of Warfare and Optimal Control. Naval Res. Log. Quart. 21, 79-106 (1974)

[36] Nolan, R.L., and Sovereign, M.G.: A Recursive Optimization and Simulation Approach to Analysis with an Application to Transportation Systems. Management Sci. 18, B-676 - B-690 (1972)

[37] Weiss, H.K.: Some Differential Games of Tactical Interest and the Value of a Supporting Weapon System. Opns. Res. 7, 180-196 (1959)

[38] Weiss, H.K.: Lanchester-Type Models of Warfare. pp. 82-98 in Proceedings of the First International Conference on Operational Research, M. Davies, R.T. Eddison, and T. Page (Editors), Operations Research Society of America, Baltimore, 1957

[39] Brackney, H.: The Dynamics of Military Combat. Opns. Res. 7, 30-44, (1959)

[40] Taylor, J.G., and Comstock, C.: Force-Annihilation Conditions for Variable-Coefficient Lanchester-Type Equations of Modern Warfare. Naval Res. Quart. 24, 349-371 (1977)

[41] Taylor, J.G.: Solving Lanchester-Type Equations for 'Modern Warfare' with Variable Coefficients. Opns. Res. 22, 756-770 (1974)

[42] Deitchman, S.J.: A Lanchester Model of Guerrilla Warfare. Opns. Res. 10, 818-827 (1962)

[43] Peterson, R.H.: Methods of Tank Combat Analysis. pp. 134-150 in "Report of Fifth Tank Conference Held at Aberdeen", H. Goldman and G. Zeller (Editors), Report No. 918, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland, July 1953 (AD 46 000)

[44] Peterson, R.H.: On the 'Logarithmic Law' of Attrition and Its Application to Tank Combat. Opns. Res. 15, 557-558 (1967)

[45] Weiss, H.K.: Combat Models and Historical Data: The U.S. Civil War. Opns. Res. 14, 759-790 (1966)

[46] Clark, G.M.: The Combat Analysis Model. Ph.D. Thesis, The Ohio State University, Columbus, Ohio, 1969

[47] Bach, R.E., Dolansky, L., and Stubbs, H.L.: Some Recent Contributions to the Lanchester Theory of Combat. Opns. Res. 10, 314-326 (1962)

[48] Morse, P.M., and Kimball, G.E.: Methods of Operations Research. The M.I.T. Press, Cambridge, Massachusetts, 1951

[49] Taylor, J.G., and Parry, S.H.: Force-Ratio Considerations for Some Lanchester-Type Models of Warfare. Opns. Res. 23, 522-533 (1975)

[50] Helmbold, R.L.: A Modification of Lanchester's Equations. Opns. Res. 13, 857-859 (1965)

[51] Helmbold, R.L.: Lanchester Parameters for Some Battles of the Last Two Hundred Years. CORG-SP-122, Combat Operations Research Group, Technical Operations, Inc., Fort Belvoir, Virginia, 1961 (AD 481 201)

[52] Helmbold, R.L.: Historical Data and Lanchester's Theory of Combat. CORG-SP-128, Combat Operations Research Group, Technical Operations, Inc., Fort Belvoir, Virginia, July 1961 (AD 480 975)

[53] Helmbold, R.L.: Historical Data and Lanchester's Theory of Combat, Part II. CORG-SP-190, Combat Operations Research Group, Technical Operations, Inc., Fort Belvoir, Virginia, August 1964 (AD 480 109)

[54] Barfoot, C.B.: The Lanchester Attrition-Rate Coefficient: Some Comments on Set Bonder's Paper and a Suggested Alternate Method. Opns. Res. 17, 888-894 (1969)

[55] Bonder, S.: The Lanchester Attrition-Rate Coefficient. Opns. Res. 15, 221-232 (1967)

[56] Bonder, S.: The Mean Lanchester Attrition Rate. Opns. Res. 18, 179-181 (1970)

[57] Kimbleton, S.R.: Attrition Rates for Weapons with Markov-Dependent Fire. Opns. Res. 19, 698-706 (1971)

[58] Cherry, W.P.: The Role of Differential Models of Combat in Fire Suppoer Analysis. Appendix 4 in Fire Support Requirements Methodology Study Phase II, Proceedings of the Fire Support Methodology Workshop, Ketron, Inc., Arlington, Virginia, December 1975

[59] Bonder, S.: A Theory for Weapon System Analysis. pp. 111-128 in Proceedings of the Fourth Annual U.S. Army Operations Research Symposium, Redstone Arsenal, Alabama, 1965

[60] Bonder, S.: A Model of Dynamic Combat. pp. IV-1 to IV-37 in Topics in Military Operations Research, The University of Michigan Engineering Summer Conferences, The University of Michigan, Ann Arbor, Michigan, August 1969

[61] Taylor, J.G.: Predicting Battle Outcome with Liouville's Normal Form for Lanchester-Type Equations of Modern Warfare. Opns. Res. 14, 185-203 (1977)

[62] Bonder, S.: The Lanchester Attrition-Rate Coefficient. Opns. Res. 15, 221-232 (1+67)

[63] Bonder, S.: The Mean Lanchester Attrition Rate. Opns. Res. 18, 179-181 (1970)

[64] Helmbold, R.L.: Decision in Battle: Breakpoint Hypotheses and Engagement Termination Data. R-772-PR, The RAND Corporation, Santa Monica, California, June 1971

[65] Department of the Army: Report of the Army Scientific Advisory Panel Ad Hoc Group on Suppression. Office of the Deputy Chief of Staff for Research, Development, and Acquisition, Washington, D.C., July 1975 (AD A017 784)

[66] Anderson, L.B., Bracken, J., Healy, J.G., Hutzler, M.J., and Kerlin, E.P.: IDA Ground-Air Model I (IDAGAM I), Volume 1: Comprehensive Description. R-199, Institute for Defense Analyses, Arlington, Virginia, October 1974

[67] Cockrell, J.K., and Ball,T.: Final Technical Report on Study to Develop an IFP Manual. Contract No. S-71-9 with Cornell Aeronautical Laboratory, Inc., The Vertex Corporation, October 1971

[68] Mulholland, R.P., and Specht, R.D.: The Rate of Advance of the Front Line in Some World War II Campaigns. RM-1072, The RAND Corporation, Santa Monica, California, April 1953

[69] Headquarters, Department of the Army: FM 105-5, Maneuver Control, Washington, D.C., December 1973

[70] Bode, J.R.: Indices of Effectiveness in General Purpose Force Analysis. BDM/W-74-070-TR, The BDM Corporation, Vienna, Virginia, October 1974

[71] Aldrich, J.R., and Bode, J.R.: The Aggregation Problem in Models of Theater-Level Warfare. BDM/W-77-129-BR, The BDM Corporation, McLean, Virginia, September 1977

[72] Howes, D.R., and Thrall, R.M.: A Theory of Ideal Linear Weights for Heterogeneous Combat Forces. Naval Res. Log. Quart. 20, 645-659 (1973)

[73] Anderson, L.B.: References on Anti-Potential Potential (The Eigenvalue Method for Computing Weapon Values). Working Paper WP-2, Project 23-31, Institute for Defense Analyses, Arlington, Virginia, March 1974

[74] Anderson, L.B.: A Briefing on Anti-Potential Potential (The Eigenvalue Method for Computing Weapon Values). Working Paper WP-2, Project 23-71, Institute for Defense Analyses, Arlington, Virginia, November 1977

[75] Emerson, D.E.: TAGS-V: A Tactical Air-Ground Warfare Model. R-1242-PR, The RAND Corporation, Santa Monica, California, June 1973

[76] Emerson, D.E.: The New TAGS Theater Air-Ground Warfare Model (Incorporating Major Ground-Combat Revisions): A Description and Operating Instructions. R-1576-PR, The RAND Corporation, Santa Monica, California, September 1974

[77] Boren, H.E., and Corwin, G.W.: CURVES: A Cost Analysis Curve-Fitting Program. R-1753-PR, The RAND Corporation, Santa Monica, California, December 1975

[78] Siska, C.P., Giamboni, L.A., and Lind, J.R.: Analytic Formulation of a Theater Air-Ground Warfare System (1953 Techniques). RM-1338, The RAND Corporation, Santa Monica, California, September 1954